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A TREATISE OF ALGEBRA.

WHEREIN
The PRINCIPLES are DEMONSTRATED,
And APPLIED
In many uſeful and intereſting ENQUIRIES, and in the RESOLUTION of a great Variety of PROBLEMS of different Kinds.

To which is added,
The GEOMETRICAL CONSTRUCTION of a great Number of LINEAR and PLANE PROBLEMS,
With the METHOD of reſolving the ſame NUMERICALLY.

K

By THOMAS SIMPSON, F.R.S.

The Sixth EDITION, REVISED.

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M.DCC.XC.

BRITISH MUSEUM

TO THE
RIGHT HONOURABLE
JAMES EARL of MORTON,
Lord ABERDOUR,
Knight of the moſt ancient Order of the THISTLE,
One of the Sixteen Peers for SCOTLAND,
Vice-Admiral of ORKNEY and ZETLAND,
Preſident of the Philoſophical Society at EDINBURGH, and
Fellow of the Royal Society of LONDON.

MY LORD,

YOUR Character will be a ſufficient apology for my deſiring the honour to inſcribe the following Sheets to your Lordſhip, and your Goodneſs will pardon the liberty I take, as it affords me an opportunity of teſtifying the high reſpect and eſteem with which I am,

MY LORD,

Your Lordſhip's moſt devoted,
moſt obedient, and moſt humble ſervant,

Thomas Simpſon.

THE

AUTHOR's PREFACE

TO THE

SECOND EDITION.

THE motives that firſt gave birth to the enſuing Work, were not ſo much any extravagant hopes the Author could form to himſelf of greatly extending the ſubject by the addition of a large variety of new improvements (though the Reader will find many things here that are no-where elſe to be met with) as an earneſt deſire to ſee a ſubject of ſuch general importance eſtabliſhed on a clear and rational foundation, and treated as a ſcience, capable of demonſtration, and not a myſterious art, as ſome authors, themſelves, have thought proper to term it.

How well the deſign has been executed, muſt be left for others to determine. It is poſſible that the pains here taken, to reduce the fundamental principles, as well as the more difficult parts of the ſubject to a demonſtration, may be looked upon, by ſome, as rather tending to throw new difficulties in the way of a Learner, than to the facilitating of his progreſs. In order to gratify, as far as might be, the inclination of this claſs of Readers, the demonſtrations are, now, given, by themſelves, in the manner of

Notes (ſo as to be taken or omitted, at pleaſure): though the Author cannot, by any means, be induced to think, that Time loſt to a Learner which is taken up in comprehending the grounds whereon he is to raiſe his ſuperſtructure: his progreſs may indeed, at firſt be a little retarded; but the *real* knowledge he thence acquires will abundantly compenſate his trouble, and enable him to proceed, afterwards, with certainty and ſucceſs, in matters of greater difficulty, where authors, and their rules, can yield him no aſſiſtance, and he has nothing to depend upon but his own obſervation and judgment.

This, ſecond, Edition has many advantages over the former, as well with reſpect to a number of new ſubjects and improvements, interſperſed throughout the whole, as in the order and diſpoſition of the elementary parts: in which particular regard has been had to the capacities of young Beginners. The Work, as it now ſtands, will, the Author flatters himſelf, be found equally plain and comprehenſive, ſo as to anſwer, alike, the purpoſe of the lower, and of the more experienced claſs of Readers.

THE CONTENTS.

SECTION I.

NOTATION Page 1

SECTION II.

ADDITION 8

SECTION III.

SUBTRACTION 11

SECTION IV.

MULTIPLICATION 13

SECTION V.

DIVISION 28

SECTION VI.

INVOLUTION 36

SECTION VII.

EVOLUTION 42

SECTION VIII.

THE REDUCTION OF FRACTIONAL, AND RADICAL QUANTITIES 45

SECTION IX.

OF EQUATIONS 57
1. *The Reduction of ſingle Equations* ibid.
2. *The Extermination of unknown Quantities, or the reduction of two or more equations to a ſingle one* 63

SECTION X.

OF ARITHMETICAL AND GEOMETRICAL PROPORTIONS 69

SECTION XI.

THE SOLUTION OF ARITHMETICAL PROBLEMS 75

SEC-

SECTION XII.

THE RESOLUTION OF EQUATIONS OF SEVERAL DIMENSIONS Page 131

1. *Of the origin and compoſition of Equations* ibid.
2. *How to know whether ſome, or all the roots of an Equation be rational, and, if ſo, what they are* 134
3. *Another way of diſcovering the ſame thing, by means of Sir* Iſaac Newton*'s method of diviſors; with the grounds and explanation of that method* 136
4. *Of the ſolution of cubic Equations according to* Cardan 143
5. *The ſame method extended to other, higher Equations* 145
6. *Of the ſolution of biquadratic Equations according to* Des Cartes 148
7. *The ſolution of biquadratics by a new method without the trouble of exterminating the ſecond term* 150
8. *Caſes of biquadratic Equations that may be reduced to quadratic ones* 153
9. *The reſolution of literal Equations, wherein the given, and the unknown quantity are alike affected* 156
10. *The reſolution of Equations by the common method of converging ſeries* 158
11. *Another way, more exact* 162
12. *A third method* 170
13. *The method of converging ſeries extended to ſurd Equations* 174
14. *A method of ſolving high Equations, when two, or more unknown quantities are concerned in each* 177

SECTION XIII.

OF INDETERMINATE PROBLEMS 180

SECTION XIV.

THE INVESTIGATION OF THE SUMS OF POWERS. 201

CONTENTS.

SECTION XV.

OF FIGURATE NUMBERS Page 213

1. *The Sums of Series, conſiſting of the reciprocals of figurate numbers, with others of the like nature* 215
2. *The ſums of compound Progreſſions, ariſing from a ſeries of powers drawn into the terms of a geometrical progreſſion* 219
3. *The combinations of Quantities* 225
4. *A demonſtration of Sir* Iſaac Newton's *Binomial theorem* 227

SECTION XVI.

OF INTEREST AND ANNUITIES 229

1. *Annuities and Penſions in Arrear, computed at ſimple intereſt* 231
2. *The inveſtigation of Theorems for the ſolution of the various caſes in compound intereſt and annuities* 234

SECTION XVII.

OF PLANE TRIGONOMETRY 241

SECTION XVIII.

THE APPLICATION OF ALGEBRA TO THE SOLUTION OF GEOMETRICAL PROBLEMS 254

1. *An eaſy way of conſtructing, or finding the roots of a quadratic Equation, geometrically* 267
2. *A demonſtration why a problem is impoſſible when the ſquare root of a negative quantity is concerned* 272
3. *A method for diſcovering whether the root of a radical quantity can be extracted* 284
4. *The manner of taking away radical quantities from the denominator of a fraction, and transferring them to the numerator* 288
5. *A method for determining the roots of certain high Equations, by means of the ſection of an angle* 301

AN APPENDIX,

Containing the geometrical conſtruction of a large variety of linear, and plane Problems; with the manner of reſolving the ſame numerically 315

20 JY 66

A TREATISE OF ALGEBRA.

SECTION I.

Of Notation.

ALGEBRA is that Science which teaches, in a general manner, the relation and compariſon of abſtract quantities: by means whereof ſuch Queſtions are reſolved whoſe ſolutions would be ſought in vain from common Arithmetic.

In Algebra, otherwiſe called *Specious Arithmetick*, Numbers are not expreſſed as in the common Notation, but every Quantity, whether given or required, is commonly repreſented by ſome letter of the alphabet; the given ones, for diſtinction ſake, being, uſually, denoted by the initial letters *a*, *b*, *c*, *d*, &c.; and the unknown, or required ones, by the final letters *u*, *w*, *x*, *y*, &c. There are, moreover, in Algebra, certain Signs or Notes made uſe of, to ſhew the relation and dependence of quantities one upon another, whoſe ſignification the Learner ought, firſt of all, to be made acquainted with.

The Sign +, *ſignifies that the quantity, which it is prefixed to, is to be added.* Thus $a + b$ ſhews that the number repreſented by b is to be added to *that* repreſented by a, and expreſſes the ſum of thoſe numbers; ſo that if a was 5, and b 3, then would $a + b$ be $5 + 3$, or 8. In like manner $a + b + c$ denotes the number ariſing by adding all the three numbers a, b, and c, together.

Note. A quantity which has no prefixed ſign (as the leading quantity a in the above examples) is always underſtood to have the ſign + before it; ſo that a ſignifies the ſame as $+ a$; and $a + b$, the ſame as $+ a + b$.

The Sign —, ſignifies that the quantity which it precedes is to be ſubtracted. Thus $a - b$ ſhews that the quantity repreſented by b is to be ſubtracted from that repreſented by a, and expreſſeth the difference of a and b; ſo that, if a was 5, and b 3, then would $a - b$ be $5 - 3$ or 2. In like manner $a + b - c - d$ repreſents the quantity which ariſes by taking the numbers c and d from the ſum of the other two numbers a and b; as, if a was 7, b 6, c 5, and d 3, then would $a + b - c - d$ be $7 + 6 - 5 - 3$, or 5.

The *Notes* + and — are uſually expreſſed by the words *plus* (or *more*) and *minus* (or *leſs*.) Thus, we read, $a + b$, *a plus b*; and $a - b$, *a minus b*.

Moreover, thoſe quantities to which the ſign + is prefixed are called *poſitive* (or *affirmative*); and thoſe to which the ſign — is prefixed, *negative*.

The Sign ×, ſignifies that the quantities between which it ſtands are to be multiplied together. Thus $a \times b$ denotes that the quantity a is to be multiplied by the quantity b, and expreſſes the product of the quantities ſo multiplied; and $a \times b \times c$ expreſſes the product ariſing by multiplying the quantities a, b, and c, continually together: thus, likewiſe, $\overline{a + b} \times c$ denotes the product of the compound quantity $a + b$ by the ſimple quantity c; and $\overline{a + b + c} \times \overline{a - b + c} \times \overline{a + c}$ repreſents the product which ariſes by multiplying the three compound quantities $a + b + c$, $a - b + c$, and $a + c$ continually together; ſo that, if a was 5, b 4, and c 3, then would $\overline{a+b+c} \times \overline{a-b+c} \times \overline{a+c}$ be 12 × 4 × 8, which is 384.

But when quantities denoted by ſingle letters are to be multiplied together, the Sign × is generally omitted, or only underſtood; and ſo ab is made to ſignify the ſame as $a \times b$; and abc, the ſame as $a \times b \times c$.

It is likewiſe to be obſerved, that when a quantity is to be multiplied by itſelf, or raiſed to any power, the uſual method of Notation is to draw a line over the given quantity, and at the end thereof place the Exponent of the Power. Thus $\overline{a+b}|^{2}$ denotes the ſame as $\overline{a+b} \times \overline{a+b}$, *viz.* the ſecond power (or ſquare) of $a+b$ conſidered as one quantity: thus, alſo, $\overline{ab+bc}|^{3}$ denotes the ſame as $\overline{ab+bc} \times \overline{ab+bc} \times \overline{ab+bc}$, *viz.* the third power (or cube) of the quantity $ab+bc$.

But in expreſſing the powers of quantities repreſented by ſingle letters, the line over the top is commonly omitted; and ſo a^2 comes to ſignify the ſame as aa or $a \times a$, and b^3 the ſame as bbb or $b \times b \times b$: whence alſo it appears, that a^2b^3 will ſignify the ſame as $aabbb$; and a^5c^2 the ſame as $aaaaacc$; and ſo of others.

The Note . (or a full point) and the word *into*, are likewiſe uſed inſtead of ×, or as Marks of Multiplication.

Thus $\overline{a+b} \,.\, \overline{a+c}$ and $\overline{a+b}$ *into* $\overline{a+c}$ both ſignify the ſame thing as $\overline{a+b} \times \overline{a+c}$, namely, the product of $a+b$ by $a+c$.

The Sign ÷ is uſed to ſignify that the quantity preceding it is to be divided by the quantity which comes after it: Thus $c \div b$ ſignifies that c is to be divided by b; and $\overline{a+b} \div \overline{a-c}$, that $a+b$ is to be divided by $a-c$.

Alſo the mark) is ſometimes uſed as a note of Diviſion; thus, $a+b)\,ab$, denotes that the quantity ab is to be divided by the quantity $a+b$; and ſo of others. But the diviſion of algebraic quantities is moſt commonly

monly expreſſed by writing down the diviſor under the dividend with a line between them (in the manner of a vulgar fraction). Thus $\frac{c}{b}$ repreſents the quantity ariſing by dividing c by b; and $\frac{a+b}{a-c}$ denotes the quantity ariſing by dividing $a+b$ by $a-c$. Quantities thus expreſſed are called algebraic fractions; whereof the upper part is called the numerator, and the lower the denominator, as in vulgar fractions.

The ſign $\sqrt{\ }$, is uſed to expreſs the ſquare root of any quantity to which it is prefixed: thus $\sqrt{25}$ ſignifies the ſquare root of 25 (which is 5, becauſe 5×5 is 25); thus alſo $\sqrt{ab}$ denotes the ſquare root of ab; and $\sqrt{\frac{ab+bc}{d}}$ denotes the ſquare root of $\frac{ab+bc}{d}$, or of the quantity which ariſes by dividing $ab+bc$ by d: but $\frac{\sqrt{ab+bc}}{d}$ (becauſe the line which ſeparates the numerator from the denominator is drawn below $\sqrt{\ }$) ſignifies that the ſquare root of $ab+bc$ is to be *firſt* taken, and *afterwards* divided by d: ſo that, if a was 2, b 6, c 4, and d 9; then would $\frac{\sqrt{ab+bc}}{d}$ be $\frac{\sqrt{36}}{9}$ or $\frac{6}{9}$; but $\sqrt{\frac{ab+bc}{d}}$ is $\sqrt{\frac{36}{9}}$, or $\sqrt{4}$, which is 2.

The ſame mark $\sqrt{\ }$, with a figure over it, is alſo uſed to expreſs the cube, or biquadratic root, &c. of any quantity: thus $\sqrt[3]{64}$ repreſents the cube root of 64, (which is 4, becauſe $4 \times 4 \times 4$ is 64), and $\sqrt[3]{ab+cd}$ the cube root of $ab+cd$; alſo $\sqrt[4]{16}$ denotes the biquadratic root of 16 (which is 2, becauſe $2 \times 2 \times 2 \times 2$ is 16); and $\sqrt[4]{ab+cd}$ denotes the biquadratic root of $ab+cd$; and ſo of others. Quantities thus expreſſed are called radical quantities, or ſurds; whereof *theſe*, conſiſting of one term only, as $\sqrt{a}$ and $\sqrt{ab}$ are called *ſimple ſurds*; and thoſe conſiſting of ſeveral terms, or members, as $\sqrt{a^2-b^2}$ and $\sqrt[3]{a^2-b^2+bc}$ *compound ſurds*.

Beſides this way of expreſſing radical quantities, (which is chiefly followed) there are other methods made uſe of by different Authors; but the moſt commodious of all, and beſt ſuited to practice, is *that* where the root is deſigned by a vulgar fraction, placed at the end of a line drawn over the quantity given. According to this Notation the ſquare root is deſigned by the fraction $\frac{1}{2}$, the cube root by $\frac{1}{3}$, and the biquadratic root by $\frac{1}{4}$, &c. Thus $\overline{a}|^{\frac{1}{2}}$ expreſſes the ſame thing with $\sqrt{a}$, *viz.* the ſquare root of a; and $\overline{a^2+ab}|^{\frac{1}{3}}$ the ſame as $\sqrt[3]{a^2+ab}$, that is, the cube root of a^2+ab: alſo $\overline{a}|^{\frac{2}{3}}$ denotes the ſquare of the cube root of a; and $\overline{a+z}|^{\frac{7}{4}}$ the ſeventh power of the biquadratic root of $a+z$; and ſo of others. But it is to be obſerved, that, when the root of a quantity repreſented by a ſingle letter is to be expreſſed, the Line over it may be neglected; and ſo $a^{\frac{1}{2}}$ will ſignify the ſame as $\overline{a}|^{\frac{1}{2}}$, and $b^{\frac{1}{3}}$ the ſame as $\overline{b}|^{\frac{1}{3}}$ or $\sqrt[3]{b}$. The number, or fraction, by which the power, or root of any quantity, is thus deſigned, is called its Index, or Exponent.

The Mark $=$ *(called the Sign of equality) is uſed to ſignify that the quantities ſtanding on each ſide of it are equal.* Thus $2+3=5$, ſhews that 2 more 3 is equal to 5; and $x=a-b$, ſhews that x is equal to the difference of a and b.

The Note $::$ *ſignifies that the quantities between which it ſtands are proportional:* As $a:b::c:d$, denotes that a is in the ſame proportion to b, as c is to d; or that if a be twice, thrice, or four times, &c. as great as b, then accordingly is c twice, thrice or four times, &c. as great as d.

To what has been thus far laid down on the ſignification of the ſigns and characters uſed in the Algebraic Notation, we may add what follows; which is equally neceſſary to be underſtood.

When any quantity is to be taken more than once, the number is to be prefixed, which ſhews how many times it is to be taken: thus $5a$ denotes that the quantity a is to be taken five times; and $3bc$ ſtands for three times bc, or the quantity which ariſes by multiplying bc by 3: alſo $7\sqrt{a^2+b^2}$ ſignifies that $\sqrt{a^2+b^2}$ is to be taken ſeven times; and ſo of others.

The numbers thus prefixed are called coefficients; and that quantity which ſtands without a coefficient is always underſtood to have an unit prefixed, or to be taken once, and no more.

Thoſe quantities are ſaid to be *like* that are expreſſed by the ſame letters under the ſame powers, or which differ only in their coefficients: thus $3bc$, $5bc$, and $8bc$ are *like* quantities; and the ſame is to be underſtood of the Radicals $2\sqrt{\frac{b+c}{a}}$ and $7\sqrt{\frac{b+c}{a}}$. But *unlike* quantities are thoſe which are expreſſed by different letters, or by the ſame letters under different powers: thus $2ab$, $2abc$, $5ab^2$ and $3ba^2$ are all *unlike*.

When a quantity is expreſſed by a ſingle letter, or by ſeveral ſingle letters joined together in multiplication (without any Sign between them) as a, or $2ab$, it is called a *ſimple* quantity.

But that quantity which conſiſts of two or more ſuch ſimple quantities, connected by the ſigns + or —, is called a *compound* quantity: thus $a-2ab+5abc$ is a *compound* quantity; whereof the *ſimple* quantities a, $2ab$ and $5abc$ are called the Terms or Members.

The letters by which any *ſimple* quantity is expreſſed may be ranged according to any order at pleaſure, and yet the ſignification continue the ſame: thus ab may be wrote ba; for ab denotes the product of a by b, and ba the product of b by a; but it is well known, that, when two numbers are to be multiplied together, it matters not which of them is made the multiplicand, nor which the multiplier, the product, either way, coming out the ſame. In like manner it will appear that *abc*, *acb*, *bac*, *bca*, *cab*, and *cba* all expreſs the ſame thing, and may be uſed indifferently for each other (as will be demonſtrated further on); but it will be ſometimes found convenient, in long operations, to place the ſeveral Letters according to the order which they obtain in the alphabet.

Likewiſe the ſeveral members, or terms of which any quantity is compoſed, may be diſpoſed according to any order at pleaſure, and yet the Signification be noways affected thereby. Thus $a-2ab+5a^2b$ may be wrote $a+5a^2b-2ab$, or $-2ab+a+5a^2b$, &c. for all *theſe* repreſent the ſame thing, *viz.* the quantity which remains, when, from the ſum of a and $5a^2b$, the quantity $2ab$ is deducted.

Here follow ſome examples wherein the ſeveral Forms of Notation hitherto explained are promiſcuouſly concerned, and where the ſignification of each is expreſſed in Numbers.

Suppoſe $a=6$, $b=5$ and $c=4$; then will

$$a^2+3ab-c^2=36+90-16=110,$$

$$2a^3-3a^2b+c^3=432-540+64=-44,$$

$$a^2\times\overline{a+b}-2abc=36\times 11-240=156,$$

$$\frac{a^3}{a+3c}+c^2=\frac{216}{18}+16=12+16=28,$$

$$\sqrt{2ac+c^2}\text{ (or }\overline{2ac+c^2}|^{\frac{1}{2}}\text{)}=\sqrt{64}=8\text{ (for }8\times 8=64\text{)}$$

$$c^{\frac{1}{2}}+\frac{2bc}{\sqrt{2ac+c^2}}=2+\frac{40}{8}=7,$$

$$\frac{a^2-\sqrt{b^2-ac}}{2a-\sqrt{b^2+ac}}=\frac{36-1}{12-7}=\frac{35}{5}=7,$$

$$\sqrt{b^2-ac}+\sqrt{2ac+c^2}=1+8=9,$$

$$\sqrt{b^2-ac+\sqrt{2ac+c^2}}=\sqrt{25-24+8}=3.$$

This method of explaining the ſignification of quantities I have found to be of good uſe to Young Beginners: And would recommend it to Such, who are deſirous of making a Proficiency in the Subject, to get a clear idea of what has been thus far delivered, before They proceed farther.

SECTION II.

Of Addition.

ADDITION, in algebra, is performed by connecting the quantities by their proper ſigns, and joining into one ſum *ſuch* as can be united: For the more ready effecting of which, obſerve the following Rules.

1°. *If, in the quantities to be added, there are Terms that are like and have all the ſame ſign, add the coefficients of thoſe terms together, and to their ſum adjoin the letters common to each term, prefixing the common ſign.*

Thus	$5a$	And	$5a+7b$	Alſo	$5a-7b$
added to	$3a$	added to	$7a+3b$	added to	$7a-3b$.
makes	$8a$.	makes	$12a+10b$.	makes	$12a-10b$.

Hence likewiſe the ſum of

$$\left\{\begin{array}{l} 2\sqrt{ab}+7\sqrt{bc} \\ 3\sqrt{ab}+2\sqrt{bc} \\ 6\sqrt{ab}+9\sqrt{bc} \end{array}\right.$$

will be $11\sqrt{ab}+18\sqrt{bc}$.

And the ſum of

$$\left\{\begin{array}{l} \frac{2b}{a}-\frac{3d}{c} \\ \frac{5b}{a}-\frac{7d}{c} \end{array}\right.$$

will be $\frac{7b}{a}-\frac{10d}{c}$

The Reaſons on which the preceding Operations are grounded will readily appear by reflecting a little on the nature and ſignification of the quantities to be added: For, with regard to the firſt example (where $3a$ is to be added to $5a$) it is plain, that three times any quantity whatever, added to five times the *ſame* quantity, muſt make eight times *that* quantity: Therefore $3a$, or three times the quantity denoted by a, being added to $5a$, or five times the ſame quantity, the ſum muſt conſequently

2°. *When, in the quantities to be added, there are like terms, whereof ſome are affirmative and others negative, add together the affirmative terms (if there be more than one) and do the ſame by the negative ones; then take the difference of the two ſums (not regarding the ſigns) by ſubtracting the coefficient of the leſſer from that of the greater and adjoining the letters common to each; to which difference prefix the ſign of the greater.*

Examples of this Rule may be as follow.

$$\begin{array}{lr} 1. & 12a - 5b \\ & -3a + 2b \\ \hline \text{Sum} & 9a - 3b \end{array} \qquad \begin{array}{lr} 2. & -3ab + 5bc \\ & +7ab - 9bc \\ \hline \text{Sum} & 4ab - 4bc \end{array}$$

$$\begin{array}{lr} 3. & 6ab + 12bc - 8cd \\ & -7ab - 9bc + 3cd \\ & -2ab - 5bc + 12cd \\ \hline \text{Sum} & -3ab - 2bc + 7cd. \end{array} \qquad \begin{array}{lr} 4. & 5\sqrt{ab} - 7\sqrt{bc} + 8d \\ & 3\sqrt{ab} + 8\sqrt{bc} - 12d \\ & 7\sqrt{ab} + 3\sqrt{bc} + 9d \\ \hline \text{Sum} & 15\sqrt{ab} + 4\sqrt{bc} + 5d. \end{array}$$

$$\begin{array}{lr} 5. & 12abc - 16abd + 25acd - 72bcd \\ & 16abc + 12abd + 20acd - 18bcd \\ & -13abc - 26abd - 15acd + 12bcd \\ & 32abc + 18abd - 10acd - 16bcd \\ \hline \text{Sum} & 47abc - 12abd + 20acd - 94bcd. \end{array}$$

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make 8*a*, or eight times that quantity. From whence, as the ſum of any two quantities is equal to the ſum of all their parts, the reaſon of the ſecond caſe, or example, is likewiſe obvious. But as to the third (where the given quantities are $5a-7b$ and $7a-3b$) we are to conſider, that, if the two quantities to be added together had been exactly 5*a* and 7*a* (which are the two leading terms) the ſum would, then, have been juſt 12*a*; but, ſince the former quantity wants 7*b* of 5*a*, and the latter 3*b* of 7*a*, their ſum muſt, it is evident, want both 7*b* and 3*b* of 12*a*; and therefore be equal to $12a-10b$, that is, equal to what remains, when the ſum of the defects is deducted. And by the very ſame way of arguing, it is eaſy to conceive that the ſum, which ariſes by adding any number of

$$\begin{array}{rl} 6. & \frac{5a}{b}-\frac{3cc}{a}+7\sqrt{\frac{bc}{a}}-9\sqrt{\frac{ab+cc}{a}} \\ & \frac{8a}{b}+\frac{7cc}{a}-12\sqrt{\frac{bc}{a}}+6\sqrt{\frac{ab+cc}{a}} \\ \hline \text{Sum} & \frac{13a}{b}+\frac{4cc}{a}-5\sqrt{\frac{bc}{a}}-3\sqrt{\frac{ab+cc}{a}} \end{array}$$

In the laſt example, and all others, where fractional and radical quantities are concerned, every ſuch quantity, excluſive of its coefficient, is to be treated in all reſpects like a ſimple quantity expreſſed by a ſingle letter.

3°. *When in the quantities to be added, there are Terms without others like to them, write them down with their proper ſigns.*

Thus $a+2b$
added to $3c+d$
makes $a+2b+3c+d$.

And $aa+bb$
added to $a+b$
makes $aa+bb+a+b$.

Here follow a few examples for the Learner's exerciſe, wherein all the three foregoing rules take place promiſcuouſly.

$$\begin{array}{rl} 1. & 2aa+3ab+8cc+d^2 \\ & 5aa-7ab+5cc-d^3 \\ & -2aa+4ab+3cc+30 \\ \hline \text{Sum} & 5aa \quad * \quad +16cc+d^2-d^3+30. \end{array}$$

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of quantities together, will be equal to the ſum of all the affirmative Terms diminiſhed by the ſum of all the negative ones (conſidered independent of their ſigns); from whence the reaſon of the ſecond general Rule is apparent. As to the caſe where the quantities are unlike, it is plain that ſuch quantities cannot be united into one, or otherwiſe added, than by their ſigns: thus, for example, let *a* be ſuppoſed to repreſent a Crown, and *b* a Shilling; then the ſum of *a* and *b* can be neither 2*a* nor 2*b*, that is, neither two crowns nor two ſhillings, but one crown *plus* on ſhilling, or *a*+*b*.

2. 5√

2. $5\sqrt{ax} - 8\sqrt{aa - xx} + 12\sqrt{aa + 4xx}$
$8\sqrt{ax} + 15\sqrt{aa - xx} - 8\sqrt{aa + 4xx}$
$6\sqrt{ax} - 7\sqrt{aa - xx} + 10\sqrt{aa + 4xx}$

Sum $19\sqrt{ax} \quad * \quad + 14\sqrt{aa + 4xx}$

3. $2a^2 - 3ab + 2b^3 - 3a^2$
$3b^3 - 2a^2 + a^3 - 5c^3$
$4c^3 - 2b^3 + 5ab + 100$
$20ab + 16a^2 - bc - 80$

Sum $13a^2 + 22ab + 3b^3 + a^3 - c^3 + 20 - bc.$

SECTION III.

Of Subtraction.

Subtraction, in Algebra, is performed by changing all the Signs of the Subtrahend (or conceiving them to be changed) and then connecting the quantities, as in addition.

Ex. 1. From $8a + 5b$
take $5a + 3b$
Rem. $3a + 2b.$

Ex. 2. From $8a + 5b$
take $5a - 3b$
Rem. $3a + 8b.$

Ex. 3. From $8a - 5b$
take $5a + 3b$
Rem. $3a - 8b.$

Ex. 4. From $8a - 5b$
take $5a - 3b$
Rem. $3a - 2b.$

In the ſecond example, conceiving the ſigns of the ſubtrahend to be changed to their contrary, *that* of $3b$ becomes +; and ſo the ſigns of $3b$ and $5b$ being *alike*, the coefficients 3 and 5 are to be added together, by caſe 1 of addition. The ſame thing happens in the third example; ſince the ſign of $3b$, when changed, is —, and therefore the ſame with that of $5b$. But, in the fourth example, the ſigns of $3b$ and $5b$, after *that* of $3b$ is changed, being *unlike*, the difference of the coefficients muſt be taken conformable to caſe 2 in addition.

Other examples in Subtraction, may be as follow.

$$\begin{array}{llcl}
\text{From} & 20ax + 5bc - 7aa & \text{From} & 7\sqrt{ax} + 9\sqrt{by} \\
\text{take} & 12ax - 3bc - 5aa & \text{take} & -5\sqrt{ax} + 12\sqrt{by} \\
\hline
\text{Rem.} & 8ax + 8bc - 2aa & \text{Rem.} & 12\sqrt{ax} - 3\sqrt{by}.
\end{array}$$

$$\begin{array}{ll}
\text{From} & 6\sqrt{aa - xx} + 10\sqrt[3]{a^3 - x^3} - 7\sqrt{\frac{aa}{c}} \\
\text{take} & 9\sqrt{aa - xx} - 15\sqrt[3]{a^3 - x^3} - 9\sqrt{\frac{aa}{c}} \\
\hline
\text{Rem.} & -3\sqrt{aa - xx} + 25\sqrt[3]{a^3 - x^3} + 2\sqrt{\frac{aa}{c}}.
\end{array}$$

$$\begin{array}{ll}
\text{From} & 7a^2 - \frac{5a}{c} + 6\sqrt{\frac{ax}{c}} + d \\
\text{take} & a^2 + \frac{8a}{c} - \sqrt{\frac{ax}{c}} + b \\
\hline
\text{Rem.} & 6a^2 - \frac{13a}{c} + 7\sqrt{\frac{ax}{c}} + d - b.
\end{array}$$

In this laſt example the quantity a^2 in the ſubtrahend, being without a coefficient, an unit is to be underſtood; for $1a^2$ and a^2 mean the ſame thing. The like is to be obſerved in all other ſimilar caſes.

The Grounds of the general rule for the ſubtraction of algebraic quantities may be explained thus: Let it be here required to ſubtract $5a-3b$ from $8a+5b$ (as in ex. 2.) It is plain, in the firſt place, that, if the affirmative part $5a$ were alone to be ſubtracted, the remainder would then be $8a+5b-5a$; but, as the quantity actually propoſed to be ſubtracted is leſs than $5a$ by $3b$, too much has been taken away by $3b$; and therefore the true remainder will be greater than $8a+5b-5a$ by $3b$; and ſo will be truly expreſſed by $8a+5b-5a+3b$: wherein the ſigns of the two laſt terms are both contrary to what they were given in the ſubtrahend; and where the whole, by uniting the like terms, is reduced to $3a+8b$, as in the example.

SECTION IV.

Of Multiplication.

BEFORE I proceed to lay down the neceſſary rules for multiplying quantities one by another, it may be proper to premiſe the following particulars, in order to give the Learner a clear idea of the reaſon and certainty of ſuch rules.

Firſt, then, it is to be obſerved, that when ſeveral quantities are to be multiplied continually together, the reſult, or product, will come out exactly the ſame, multiply them according to what order you will. Thus $a \times b \times c$, $a \times c \times b$, $b \times c \times a$, *&c.* have all the ſame value, and may be uſed indifferently: To illuſtrate which we may ſuppoſe $a = 2$, $b = 3$, and $c = 4$; then will $a \times b \times c = 2 \times 3 \times 4 = 24$; $a \times c \times b = 2 \times 4 \times 3 = 24$; and $b \times c \times a = 3 \times 4 \times 2 = 24$.

Secondly. *If any number of quantities be multiplied continually together, and any other number of quantities be alſo multiplied continually together, and then the two products one into the other, the quantity thence ariſing will be equal to the quantity that ariſes by multiplying all the propoſed quantities continually together.* Thus will $abc \times de = a \times b \times c \times d \times e$; ſo that, if a was $= 2$, $b = 3$, $c = 4$, $d = 5$, $e = 6$, then would $abc \times de = 24 \times 30 = 720$, and $a \times b \times c \times d \times e = 2 \times 3 \times 4 \times 5 \times 6 = 720$. The general Demonſtration of theſe obſervations is given below in the notes.

The following Demonſtrations depend on this Principle, *that if two quantities, whereof the one is n times as great as the other (n being any number at pleaſure), be multiplied by one and the ſame quantity, the product, in the one caſe, will alſo be n times as great as in the other.* The greater quantity may be conceived to be divided into n parts, equal, each, to the leſſer quantity; and the product of each part (by the given multiplier) will be

The multiplication of algebraic quantities may be conſidered in the ſeven following caſes.

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be equal to that of the ſaid leſſer quantity; therefore the ſum of the products of all the parts, which make up the whole greater product, muſt neceſſarily be n times as great as the leſſer product, or the product of one ſingle part, alone.

This being premiſed, it will readily appear, in the firſt place, that $b \times a$ and $a \times b$ are equal to each other: For, $b \times a$ being b times as great as $1 \times a$ *(becauſe the multiplicand is b times as great)* it muſt therefore be equal to $1 \times a$ (or a), repeated b times, that is, equal to $a \times b$, *by the definition of multiplication.*

In the ſame manner, the equality of all the variations, or products, *abc*, *bac*, *acb*, *cab*, *bca*, *cba* (where the number of factors is 3) may be inferred: For thoſe that have the laſt factor the ſame *(which I call of the ſame claſs)* are manifeſtly equal, being produced of equal quantities multiplied by the ſame quantity: And, to be ſatisfied that thoſe *of different claſſes*, as *abc* and *acb*, are likewiſe equal, we need only conſider, that, ſince $ac \times b$, is c times as great as $a \times b$ (becauſe the multiplicand is c times as great) it muſt therefore be equal to $a \times b$ taken c times, that is, equal to $a \times b \times c$, *by the definition of multiplication.*

Univerſally. If all the Products, when the number of factors is n, be equal, all the Products, when the number of factors is $n + 1$, will likewiſe be equal: For thoſe *of the ſame claſs* are equal, being produced of equal quantities multiplied by the ſame quantity: and, to ſhew that thoſe *of different claſſes* are equal alſo, we need only take two Products which differ in their two laſt factors, and have all the preceding ones according to the ſame order, and prove them to be equal. Theſe two factors we will ſuppoſe to be repreſented by r and s, and the Product of all the preceding ones by p; then the two Products themſelves will be repreſented by prs and psr, which are equal, *by caſe* 2.

Thus,

1°. *Simple quantities are multiplied together by multiplying the coefficients one into the other, and to the product annexing the quantity which, according to the method of notation, expreſſes the product of the ſpecies; prefixing the ſign + or —, according as the ſigns of the given quantities are like or unlike.*

Thus	$2a$	Alſo	$6ab$	And	$11adf$
mult. by	$3b$	mult. by	$5c$	mult. by	$7ab$
makes	$6ab$.	makes	$30abc$.	makes	$77aabdf$.

Thus, by way of illuſtration, $abcde$ will appear to be $= abced$, &c. For, the former of theſe being equal to every other product of the claſs, or termination e (by hypotheſis and equal multiplication), and the latter equal to every other Product of the claſs, or termination d; it is evident, therefore, that all the Products of different claſſes, as well as of the ſame claſs, are mutually equal to each other.

So far relates to the firſt general obſervation: It remains to prove that $abcd \times pqrst$ is $= a \times b \times c \times d \times p \times q \times r \times s \times t$. In order to which, let $abcd$ be denoted by x, then will $abcd \times pqrst$ be denoted by $x \times pqrst$, or $pqrst \times x$ (by caſe 1), that is, by $p \times q \times r \times s \times t \times x$; which is equal to $x \times p \times q \times r \times s \times t$, or $a \times b \times c \times d \times p \times q \times r \times s \times t$, *by the preceding Demonſtration.*

The Reaſon of Rule 1 depends on theſe two general Obſervations: for it is evident from hence, that $2a \times 3b$ (in the firſt example) is $= 2 \times a \times 3 \times b = 2 \times 3 \times a \times b = 6 \times a \times b = 6ab$: And, in the ſame manner, $11adf \times 7ab$ (in the third example) appears to be $= 11 \times a \times d \times f \times 7 \times a \times b = 11 \times 7 \times a \times a \times b \times d \times f = 77 \times aabdf = 77aabdf$. But the grounds of the method of proceeding may be otherwiſe explained, thus: It has been obſerved that ab (according to the method of notation) defines the product of the Species a, b (in the firſt example), therefore the product of a by $3b$, which muſt be three times as great (becauſe the multiplier is here three times as great), In the preceding examples all the products are *affirmative*, the quantities given to be multiplied being ſo; but, in thoſe that follow, ſome are *affirmative*, and others *negative*, according to the different caſes ſpecified in the latter part of the rule; whereof the reaſons will be explained hereafter.

$$\begin{array}{llll}
\text{Mult.} & +\ 5a & \text{Mult.} & -\ 5a & \text{Mult.} & -\ 5a \\
\text{by} & \underline{-\ 6b} & \text{by} & \underline{+\ 6b} & \text{by} & \underline{-\ 6b} \\
\text{Prod.} & -\ 30ab. & \text{Prod.} & -\ 30ab. & \text{Prod.} & +\ 30ab.
\end{array}$$

$$\begin{array}{llll}
\text{Mult.} & +\ 7\sqrt{ax} & \text{Mult.} & -\ 7a\sqrt{aa+xx} \\
\text{by} & \underline{-\ 5\sqrt{cy}} & \text{by} & \underline{-\ 6b\sqrt{aa-yy}} \\
\text{Prod.} & -35\times\sqrt{ax}\times\sqrt{cy}. & \text{Prod.} & +42ab\times\sqrt{aa+xx}\times\sqrt{aa-yy}
\end{array}$$

In the two laſt examples, and all others, where radical quantities are concerned, every ſuch quantity may be conſidered, and treated in all reſpects as a ſimple quantity, expreſſed by a ſingle letter; ſince it is not the Form of the expreſſion, but the value of the quantity that is here regarded.

2°. *A Fraction is multiplied, by multiplying the numerator thereof by the given multiplier, and making the product a numerator to the given denominator,*

Thus $\frac{a}{b}\times c$ makes $\frac{ac}{b}$; alſo $\frac{3ac}{b}\times 2ad$ makes $\frac{6aacd}{b}$;

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will be truly defined by $3ab$, or ab taken three times: but, ſince the product of a by $3b$ appears to be $3ab$, it is plain that the product of $2a$ by $3b$ muſt be twice as great as *that* of a by $3b$, and therefore will be truly expreſſed by $6ab$. Thus alſo, the product of the Species ab and c (in the ſecond example) being abc (by bare notation) it is evident that the product of $6ab$ by c will be truly defined by $6abc$, or abc ſix times taken, and conſequently the product of $6ab$ and $5c$, by $30abc$, or $6abc$ taken five times, the multiplier here being five times as great.

The Reaſon of Rule 2° may be thus demonſtrated: Let the numerator of any propoſed fraction be denoted by A, the likewiſe $\frac{2ab}{c+d} \times 7\sqrt{ax}$ makes $\frac{14ab\sqrt{ax}}{c+d}$; laſtly $\frac{5ab}{\sqrt{aa+xx}}$ $\times 2ab$ makes $\frac{10a^2b^2}{\sqrt{aa+xx}}$.

3°. *Fractions are multiplied into one another, by multiplying the numerators together for a new numerator, and the denominators together for a new denominator.*

Thus, $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$; $\frac{2ab}{3c} \times \frac{5ad}{3f} = \frac{10a^2bd}{9cf}$; $\frac{21xy}{\sqrt{az}} \times \frac{3a\sqrt{a}}{8b} = \frac{63axy\sqrt{a}}{8b\sqrt{az}}$; $\frac{-5a\sqrt{x}}{3bc} \times \frac{-2a}{b} = \frac{10a^2\sqrt{x}}{3bbc}$; and $\frac{3a\sqrt{xy}}{\sqrt{ab}} \times \frac{5b\sqrt{aa+xx}}{a+z} = \frac{15ab \times \sqrt{xy} \times \sqrt{aa+xx}}{\overline{a+z} \times \sqrt{ab}}$.

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the denominator by B, and the given multiplicator by C: then, I ſay, that $\frac{AC}{B}$ is equal to $\frac{A}{B} \times C$. For ſince $\frac{AC}{B}$ denotes the quantity which ariſes by dividing AC by B, and $\frac{A}{B}$ the quantity which ariſes by dividing A by B, it is evident that the former of theſe two quantities muſt be C times as great as the latter (becauſe the dividual is C times as great in the one caſe as in the other) and therefore muſt be equal to the latter C times taken, that is, $\frac{AC}{B}$ muſt be equal to $\frac{A}{B} \times C$, as was to be ſhewn.

The Reaſon of Rule 3° will appear evident from the preceding demonſtration of Rule 2°. For, it being there proved that $\frac{A}{B} \times C$ is equal to $\frac{AC}{B}$, it is obvious that $\frac{A}{B} \times \frac{C}{D}$ can be only the D part of $\frac{AC}{B}$; be-

4°. *Surd quantities under the ſame radical ſign are multiplied like rational quantities, only the product muſt ſtand under the ſame radical ſign.*

Thus, $\sqrt{7} \times \sqrt{5} = \sqrt{35}$; $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$; $\sqrt[3]{7bc} \times \sqrt[3]{5ad} = \sqrt[3]{35abcd}$; $3\sqrt{ab} \times 5\sqrt{c} = 15\sqrt{abc}$; $2a\sqrt{2cy} \times 3b\sqrt{5ax}$ $(= 6ab \times \sqrt{2cy} \times \sqrt{5ax})$ $= 6ab\sqrt{10acxy}$; and $\frac{7ab}{5x}\sqrt{\frac{8x}{3a}} \times \frac{5c}{9d}\sqrt{\frac{13d}{2b}} = \frac{35abc}{45dx}\sqrt{\frac{104dx}{6ab}}$.

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cauſe, $\frac{C}{D}$, the multiplier here, is but the D part of the former multiplier C: But $\frac{AC}{BD}$ is alſo equal to the D part of the ſame $\frac{AC}{B}$; becauſe its diviſor is D times as great as that of $\frac{AC}{B}$: therefore theſe two quantities, $\frac{A}{B} \times \frac{C}{D}$ and $\frac{AC}{BD}$ being the ſame part of one and the ſame quantity, they muſt neceſſarily be equal to each other; *which was to be proved.*

As to Rule 4° for the multiplication of ſimilar radical quantities, it may be explained thus: Suppoſe $\sqrt{A}$ and $\sqrt{B}$ to repreſent the two given quantities to be multiplied together; let the former of them be denoted by a, and the latter by b, that is, let the quantities repreſented by a and b be ſuch, that aa may be $=$ A, and $bb = B$; then the product of $\sqrt{A}$ by $\sqrt{B}$, or of a by b, will be expreſſed by ab, and its ſquare by $ab \times ab$: but $ab \times ab$ is $= a \times b \times a \times b = aa \times bb$ (by the general obſervations premiſed at the beginning of this ſection); whence the ſquare of the product is likewiſe truly expreſſed by $aa \times bb$, or its equal $A \times B$; and conſequently the product itſelf, by $\sqrt{A \times B}$, that is, by the quantity which, being multiplied into itſelf, produces $A \times B$.

In

5°. *Powers, or roots of the ſame quantity are multiplied together, by adding their exponents:* But the exponents here underſtood are thoſe defined in p. 5, where roots are repreſented as fractional powers.

Thus, $x^2 \times x^3$ is $= x^5$; $\overline{a+z}|^{3} \times \overline{a+z}|^{5} = \overline{a+z}|^{8}$; $x^2 \times x^{\frac{1}{2}} = x^{2+\frac{1}{2}} = x^{\frac{5}{2}}$; and $x^{\frac{1}{2}} \times x^{\frac{1}{2}} = x^1 = x$; alſo $\overline{aa+zz}|^{\frac{2}{3}} \times \overline{aa+zz}|^{\frac{1}{3}}$ is $= \overline{aa+zz}|^{1} = aa+zz$; and $\overline{c+y}|^{\frac{1}{2}} \times \overline{c+y}|^{\frac{1}{3}} = \overline{c+y}|^{\frac{1}{2}+\frac{1}{3}} = \overline{c+y}|^{\frac{5}{6}}$ *.

In the ſame manner the product of $\sqrt[3]{\text{A}} \times \sqrt[3]{\text{B}}$ will appear to be $\sqrt[3]{\text{AB}}$: for, if $\sqrt[3]{\text{A}}$ be denoted by a, and $\sqrt[3]{\text{B}}$ by b; or, which is the ſame, if $aaa = \text{A}$, and $bbb = \text{B}$; then will $\sqrt[3]{\text{A}} \times \sqrt[3]{\text{B}} = a \times b$ (or ab) and its cube $= ab \times ab \times ab = aaa \times bbb = \text{AB}$ (by the aforeſaid obſervations) whence the product itſelf will evidently be expreſſed by $\sqrt[3]{\text{AB}}$.

* The Grounds of theſe Operations may be thus explained. Firſt, when the exponents are whole numbers, as in example 1, the demonſtration is obvious, from the general obſervations premiſed at the beginning of the ſection: For, by what is there ſhewn, $x^2 \times x^3$, or $xx \times xxx$ is $= x \times x \times x \times x \times x = x^5$ (*by Notation.*) But in the laſt example, where the exponents are fractions, let $\overline{c+y}|^{\frac{1}{6}}$ be repreſented by x; that is, let the quantity x be ſuch, that $x \times x \times x \times x \times x \times x$, or x^6 may be equal to $c+y$; ſo ſhall $\overline{c+y}|^{\frac{1}{2}}$ be expreſſed by x^3; becauſe, by what has been already ſhewn, $x^3 \times x^3$ is $= x^6$: and, in the ſame manner, will $\overline{c+y}|^{\frac{1}{3}}$ be expreſſed by x^2; becauſe $x^2 \times x^2 \times x^2$ is likewiſe $= x^6$. Therefore $\overline{c+y}|^{\frac{1}{2}} \times \overline{c+y}|^{\frac{1}{3}}$ is $= x^3 \times x^2 = x^5 =$ the fifth power of $\overline{c+y}|^{\frac{1}{6}}$; which is $\overline{c+y}|^{\frac{5}{6}}$, *by Notation.*

6°. *A Compound quantity is multiplied by a ſimple one, by multiplying every term of the multiplicand by the multiplier.*

Thus $a+2b-3c$ Alſo $a^2-5a\sqrt{x}+7b$
mult. by $3a$ mult. by $8c$
makes $3a^2+6ab-9ac$; makes $8a^2c-40ac\sqrt{x}+56bc$;
And $5a^2-8ab+6ac-7bc+12b^2-9c^2$
mult. by $3abc$
makes $15a^3bc-24a^2b^2c+18a^2bc^2-21ab^2c^2+36ab^3c-27abc^3$.

To explain the Reaſon of the two laſt Rules, let it be, *firſt*, propoſed to multiply any compound quantity, as $a+b-c-d$, by any ſimple quantity f; and, I ſay, the product will be $af+bf-cf-df$. For, the product of the affirmative terms, $a+b$, will be $af+bf$, becauſe, to multiply one quantity by another, is to take the multiplicand as many times as there are units in the multiplier, and to take the whole multiplicand $(a+b)$ any number of times (f), is the ſame as to take all its parts (a, b) the ſame number of times, and add them together. Moreover, ſeeing $a+b-c-d$ denotes the exceſs of the affirmative terms (a and b) above the negative ones (c and d,) therefore, to multiply $a+b-c-d$ by f, is only to take the ſaid exceſs f times; but f times the exceſs of any quantity above another is, manifeſtly, equal to f times the former quantity, *minus* f times the latter; but f times the former is, here, equal to $af+bf$ (by what has been already ſhewn) and f times the latter (for the ſame reaſon) will be equal to $cf+df$, and therefore the product of $a+b-c-d$ by f, is equal to $af+bf-cf-df$; as was to be proved. Hence it appears, that a compound quantity is multiplied by a ſimple affirmative quantity, by multiplying every term of the former by the latter, and connecting the terms thence ariſing with the ſigns of the multiplicand.

But,

7° *Compound quantities are multiplied into one another, by multiplying every term of the multiplicand by each term of the multiplier, ſucceſſively, and connecting the ſeveral products thus ariſing with the ſigns of the multiplicand, if the multiplying term be affirmative, but with contrary ſigns, if negative.*

Thus the product of $5a + 3x$
multiplied by $\underline{3a + 2x}$
will be $\left\{\begin{array}{l} 15aa + 9ax \\ \qquad\quad + 10ax + 6xx \end{array}\right\}$
which, contracted by uniting the like terms, is $\Big\}\ 15aa + 19ax + 6xx$.

But, to prove that the Method alſo holds when both the quantities are compound ones, let it be, now, propoſed to multiply A—B by C—D; then, I ſay, the product will be truly expreſſed by AC—BC—AD+BD. For, it has been already obſerved, that to multiply one quantity by another, is to take the multiplicand as many times as there are units in the multiplier; and therefore, to multiply A—B by C—D is only to take A—B as many times as there are units in C—D: Now (according to the method of multiplying compound quantities) I firſt take A—B, C times (or multiply by C) and the quantity thence ariſing will be AC—BC *(by what is demonſtrated above)*. But, I was to have taken A—B only C—D times; therefore, by this firſt Operation, I have taken *it* D times too much; whence, to have the true product, I ought to deduct D times A—B from AC—BC, the quantity thus found; but D times A—B *(by what is already proved)* is equal to AD—BD; which ſubtracted from AC—BC, or wrote down with its ſigns changed, gives the true product, AC—BC—AD+BD, *as was to be demonſtrated.* And, *univerſally*, if the ſign of any propoſed term of the multiplier, in any caſe whatever, be affirmative, it is eaſy to conceive that the required product will be greater than it would be if there

Likewiſe the product

of $a^3 + a^2b + ab^2 + b^3$

by $a - b$

is $$\left\{\begin{array}{l} a^4 + a^3b + a^2b^2 + ab^3 \\ \quad - a^3b - a^2b^2 - ab^3 - b^4 \end{array}\right\}$$

Which, by ſtriking out the terms that deſtroy one another, becomes $a^4 - b^4$.

---

were no ſuch term, by the product of that term into the whole multiplicand; and therefore it is, that this product is to be added, or wrote down with its proper ſigns, which are proved above to be thoſe of the multiplicand. But if, on the contrary, the ſign of the term, by which you multiply, be negative; then, as the required product muſt be leſs than it would be, if there were no ſuch term, by the product of that term into the whole multiplicand, this product, it is manifeſt, ought to be ſubtracted, or wrote down with contrary ſigns.

Hence is derived the common Rule, *that like Signs produce* +, *and unlike Signs* —.

For, firſt, if the ſigns of both the quantities, or terms, to be multiplied are affirmative (and therefore *like*) it is plain that the ſign of the product muſt likewiſe be affirmative.

Secondly, alſo if the ſigns of both quantities are negative (and therefore ſtill *like*) that of the product will be affirmative, *becauſe contrary to that of the multiplicand, by what has been juſt now proved.*

Thirdly, but if the ſign of the multiplicand be affirmative, and that of the multiplier negative, (and therefore *unlike*) the ſign of the product will be negative, *becauſe contrary to that of the multiplicand.*

Laſtly, if the ſign of the multiplicand be negative, and that of the multiplier affirmative, (and therefore ſtill *unlike*) the ſign of the product will be negative, *becauſe the ſame with that of the multiplicand.*

And theſe four are all the Caſes that can poſſibly happen with regard to the variation of ſigns.

Other examples in Multiplication, for the Learner's exerciſe, may be as follow; from which he may (if he pleaſes) proceed directly to Diviſion, by paſſing over the intervening Scholium.

$$
\begin{array}{llcccccccc}
\text{1. Multiply} & x^2 & + & xy & + & y^2 & & & & \\
\text{by} & x^2 & - & xy & + & y^2 & & & & \\
\hline
 & x^4 & + & x^3y & + & x^2y^2 & & & & \\
 & & - & x^3y & - & x^2y^2 & - & xy^3 & & \\
 & & & & + & x^2y^2 & + & xy^3 & + & y^4 \\
\hline
\text{product} & x^4 & & * & + & x^2y^2 & & * & + & y^4.
\end{array}
$$

$$
\begin{array}{llcccccccc}
\text{2. Multiply} & 2a^2 & - & 3ax & + & 4x^2 & & & & \\
\text{by} & 5a^2 & - & 6ax & - & 2x^2 & & & & \\
\hline
 & 10a^4 & - & 15a^3x & + & 20a^2x^2 & & & & \\
 & & - & 12a^3x & + & 18a^2x^2 & - & 24ax^3 & & \\
 & & & & - & 4a^2x^2 & + & 6ax^3 & - & 8x^4 \\
\hline
\text{product} & 10a^4 & - & 27a^3x & + & 34a^2x^2 & - & 18ax^3 & - & 8x^4.
\end{array}
$$

$$
\begin{array}{llcccccccccc}
\text{3. Multiply} & 3a & - & 2b & + & 2c & & & & & & \\
\text{by} & 2a & - & 4b & + & 5c & & & & & & \\
\hline
 & 6aa & - & 4ab & + & 4ac & & & & & & \\
 & & - & 12ab & + & 8bb & - & 8bc & & & & \\
 & & & & + & 15ac & - & 10bc & + & 10cc & & \\
\hline
\text{product} & 6aa & - & 16ab & + & 19ac & + & 8bb & - & 18bc & + & 10cc.
\end{array}
$$

$$
\begin{array}{llcccccccccc}
\text{4. Multiply} & a^3 & - & 3a^2b & + & 3ab^2 & - & b^3 & & & & \\
\text{by} & a^2 & - & 2ab & + & b^2 & & & & & & \\
\hline
 & a^5 & - & 3a^4b & + & 3a^3b^2 & - & a^2b^3 & & & & \\
 & & - & 2a^4b & + & 6a^3b^2 & - & 6a^2b^3 & + & 2ab^4 & & \\
 & & & & + & a^3b^2 & - & 3a^2b^3 & + & 3ab^4 & - & b^5 \\
\hline
\text{product} & a^5 & - & 5a^4b & + & 10a^3b^2 & - & 10a^2b^3 & + & 5ab^4 & - & b^5.
\end{array}
$$

SCHOLIUM.

The manner of proceeding in referring the reaſons of the different caſes of the ſigns to the multiplication of compound quantities, may perhaps be looked upon as indirect, and contrary to good method; according to which, it may be thought, that theſe reaſons ought to have been given before, along with the rules for ſimple quantities, as it is the way that almoſt all Authors on the ſubject have followed.

But, however indirect the method here purſued may ſeem, it appears to me the moſt clear and rational; and I believe it will be found very difficult, if not impoſſible, without explaining the rules for compound quantities firſt, to give a Learner a *diſtinct* Idea how the product of two ſimple quantities, with negative ſigns, ſuch as $-b$ and $-c$, ought to be expreſſed, when they ſtand alone, independent of all other quantities: And I cannot help thinking farther, that the difficulties about the ſigns, ſo generally complained of by Beginners, have been much more owing to the manner of explaining them, this way, than to any real intricacy in the ſubject itſelf; nor will this opinion, perhaps, appear ill grounded, if it be conſidered that both $-a$ and $-b$, as they ſtand here independently, are as much impoſſible in one ſenſe, as the imaginary ſurd quantities $\sqrt{-b}$ and $\sqrt{-c}$; ſince the ſign $-$, according to the eſtabliſhed Rules of notation, ſhews that the quantity to which *it* is prefixed, is to be ſubtracted; but, to ſubtract ſomething from nothing is impoſſible, and the notion, or ſuppoſition of a quantity leſs than nothing, abſurd and ſhocking to the imagination: And, certainly, if the matter be viewed in this light, it would be very ridiculous to pretend to prove, by any *ſhew* of reaſoning, what the product of $-b$ by $-c$, or of $\sqrt{-b}$ by $\sqrt{-c}$, muſt be, when we can have *no Idea* of the value of the quantities to be multiplied. If, indeed, we were to look upon $-b$ and $-c$ as *real* quantities, the ſame as repreſented to the mind by b and c (which cannot be done conſiſtently, in pure Algebra, where magnitude *only* is regarded), we might then attempt to explain the matter in the ſame manner that ſome others have done; from the conſideration, *that*, as the ſign $-$ is oppoſite in its nature to the ſign $+$, *it* ought therefore to have in all operations an oppoſite effect; and conſequently, that as the product, when the ſign + is prefixed to the multiplier, is to be added; ſo, on the contrary, the product, when the ſign — is prefixed, ought to be ſubtracted.

But this way of arguing, however reaſonable it may appear, ſeems to carry but very little of ſcience in *it*, and to fall greatly ſhort of the evidence and conviction of a demonſtration: nay, *it* even claſhes with Firſt Principles, and the more eſtabliſhed Rules of notation; according to which the ſigns + and — are relative only to the magnitudes of quantities, as compoſed of different terms or members, and not to any future operations to be performed by them: Beſides, when we are told that the product ariſing from a negative multiplier is to be ſubtracted, we are not told what it is to be ſubtracted from; nor is there any thing from whence it *can be* ſubtracted, when negative quantities are independently conſidered. And farther, to reaſon about oppoſite effects, and recur to ſenſible objects and popular conſiderations, ſuch as debtor and creditor, *&c.* in order to demonſtrate the principles of a ſcience whoſe Object is abſtract Number, appears to me, not well ſuited to the nature of ſcience, and to derogate from the dignity of the ſubject.

It muſt be allowed, that in the application of Algebra to different branches of mixed mathematics, where the conſideration of oppoſite qualities, effects, or poſitions can have place, the uſual methods have a better foundation; and the conception of a quantity abſolutely negative becomes leſs difficult. Thus, for example, a line may be conceived to be produced out, both ways, from any point aſſigned; and the part on the one ſide of that point being taken as *poſitive*, the other will be *negative*. But the caſe is not the ſame in abſtract Number; whereof the beginning is fixed in the nature of things, from whence we can proceed only one way.

There can, therefore, be no ſuch things as negative numbers, or quantities abſolutely negative in pure Algebra, whoſe Object is Number, and where every multiplication, diviſion, *&c.* is a multiplication, diviſion,

ſion, &c. of Numbers, even in the application thereof: For, when we reaſon upon the quantities *themſelves*, and not upon the *numbers* expreſſing the meaſures of *them*, the proceſs becomes *purely geometrical*, whatever ſymbols may be uſed therein, from the algebraic notation; which can be of no other uſe here than to abbreviate the work.

However, after all, it may be neceſſary to ſhew upon what kind of evidence the multiplication of negative, and imaginary quantities is grounded, as theſe ſometimes occur, in the reſolution of problems: In order to which it will be requiſite to obſerve, that, as all our reaſoning regards *real*, *poſitive* quantities, ſo the algebraic expreſſions, whereby ſuch quantities are exhibited, muſt likewiſe be real and poſitive. But, when the problem is brought to an equation, the caſe may indeed be otherwiſe; for, in ordering the equation, ſo much may be taken away from both ſides thereof, as to leave the remaining quantities negative; and then it is, chiefly, that the multiplication by quantities abſolutely negative takes place.

Thus if there were given the equation $a - \frac{x}{b} = c$ (in order to find x); then by ſubtracting the quantity a from each ſide thereof, we ſhall have $-\frac{x}{b} = c - a$; which multiplied by $-b$, *according to the general Rule*, gives $x = -cb + ab$; that is $-\frac{x}{b}$ by $-b$ will give $+x$; c by $-b$, $-cb$; and $-a$ by $-b$, $+ab$; which appear to be true; becauſe the products being thus expreſſed, the ſame concluſion is derived, as if both ſides of the original equation had been firſt increaſed by $\frac{x}{b} - c$, and then multiplied by b; where both the multiplier and multiplicand are real, affirmative quantities, and where the whole operation is, therefore, capable of a clear and ſtrict demonſtration: but then, it is not in conſequence of any reaſoning I am capable of forming about $-\frac{x^2}{b}$ and $-b$, or about $+c$ and $-b$, conſidered independently, that I can be *certain* that their product ought to be expreſſed in that manner.

So likewiſe, if there were given the equation $a - \frac{x^2}{b} = c$; by tranſpoſing a and taking the ſquare root on both ſides, we ſhall have $\sqrt{-\frac{x^2}{b}} = \sqrt{c-a}$; and this multiplied by $\sqrt{-b}$, will give $\sqrt{x^2}$ (or x) $= \sqrt{-cb+ab}$: which alſo appears to be true, becauſe the reſult, this way, comes out exactly the ſame, as if the operations, for finding x, had been performed altogether by *real* quantities: But, notwithſtanding this, it is not from any reaſoning that I can form, about the multiplication of the *imaginary* quantities $\sqrt{-\frac{x^2}{b}}$ and $\sqrt{-b}$, &c. conſidered independently, that I can prove their product ought to be ſo expreſſed; for it would be very abſurd to pretend to demonſtrate what the product of two expreſſions muſt be, which are impoſſible in themſelves, and of whoſe values we can form no idea. It indeed ſeems reaſonable, that the known rules for the ſigns, as they are proved to hold in all caſes whatever, where it is poſſible to form a demonſtration, ſhould alſo anſwer *here*: But the ſtrongeſt evidence we can have of the truth and certainty of concluſions derived by means of negative and imaginary quantities, is, the exact, and conſtant agreement of ſuch concluſions with thoſe determined from more demonſtrable methods wherein no ſuch quantities have place.

In the foregoing conſiderations, the negative quantities $-b$, $-c$, &c. have been repreſented, in ſome caſes, as a kind of imaginary, or impoſſible quantities; it may not, therefore, be improper to remark here, that ſuch imaginary quantities ſerve, many times, as much to diſcover the impoſſibility of a problem, as imaginary ſurd quantities: for it is plain, that, in all queſtions relating to abſtract Numbers, or ſuch wherein magnitude *only* is regarded, and where no conſideration of poſition, or contrary values, can have place; I ſay, in all ſuch caſes, it is plain that the ſolution will be altogether as impoſſible, when the concluſion comes out a negative quantity, as if it were actually affected with an imaginary ſurd; ſince, in the one caſe, it is required that a number ſhould be actually leſs than nothing; and in the other, that the double rectangle of two numbers ſhould be greater than the ſum of their ſquares; both which are equally impoſſible: But, as an inſtance of the impoſſibility of ſome ſort of queſtions, when the concluſion comes out negative, let there be given, in a right-angled Triangle, the ſum of the hypothenuſe and perpendicular $= a$, and the baſe $= b$, to find the perpendicular; then (by what ſhall hereafter be ſhewn in its proper place) the Anſwer will come out $\frac{a^2 - b^2}{2a}$, and is poſſible, or impoſſible, according as the quantity $\frac{a^2 - b^2}{2a}$ is affirmative or negative, or as a is greater or leſs than b; which will manifeſtly appear from a bare contemplation of the problem: and the ſame thing might be inſtanced in a variety of other examples.

SECTION V.

Of Diviſion.

Diviſion in ſpecies, as in numbers, is the converſe of multiplication, and is comprehended in the ſeven following caſes.

1°. *When one ſimple quantity is to be divided by another, and all the factors of the diviſor are alſo found in the dividend, let thoſe factors be all caſt off or expunged, then the remaining factors of the dividend, joined together, will expreſs the quotient ſought.* But it is to be obſerved that, both here and in the ſucceeding caſes, the ſame rule is to be regarded in relation to the ſigns, as in multiplication, *viz.* that *like Signs give* +, *and unlike* —. It may alſo be proper to obſerve, that, when any quantity is to be divided by itſelf, or an equal quantity, the quotient will be expreſſed by an unit, or 1.

Thus $a \div a$, gives 1; and $2ab \div 2ab$ gives 1; moreover $3abcd \div ac$, gives $3bd$; and $16bc \div 8b$, gives $2c$: for the dividend here, by reſolving its coefficient into two factors, becomes $2 \times 8 \times b \times c$; from whence caſting off 8 and b, thoſe common to the diviſor, we have $2 \times c$, or $2c$. In the ſame manner, by reſolving, or dividing the coefficient of the dividend by *that* of the diviſor, the quotient will be had in other caſes: Thus, $20abc$ divided by $4c$, gives $5ab$; and $-51ab\sqrt{xy} \times \sqrt{xx + yy}$, divided by $-17a\sqrt{xy}$, gives $+3b\sqrt{xx + yy}$.

2°. *But if all the factors of the diviſor are not to be found in the dividend, caſt off thoſe (if any ſuch there be) that are common to both, and write down the remaining factors of the diviſor, joined together, as a denominator to thoſe of the dividend; ſo ſhall the fraction thus ariſing expreſs the quotient ſought.* But if, by proceeding thus, all the factors in the dividend ſhould happen to go off, or vaniſh, then an unit will be the numerator of the fraction required.

Thus, abc divided by bcd, gives $\frac{a}{d}$:

And $16a^2bx^3$ divided by $8abcx^2$, gives $\frac{2ax}{c}$:

The firſt Rule, given above, being exactly the converſe of Rule 1° in the preceding ſection, requires no other demonſtration than is there given. The ſecond Rule (as well as thoſe that follow hereafter upon Fractions) depend on this principle, That, as many times as any one propoſed quantity is contained in another, juſt ſo many times is the half, third, fourth, or any other aſſigned part of the former, contained in the half, third, fourth, or other correſponding part of the latter; and juſt

Likewiſe $27ab\sqrt{xy}$ divided by $9a^2\sqrt{xy}$, gives $\frac{3b}{a}$;

And $8ab\sqrt{ay}$ divided by $16a^2b\sqrt{ay}$, gives $\frac{1}{2a}$.

3°. *One Fraction is divided by another, by multiplying the denominator of the diviſor into the numerator of the dividend for a new numerator, and the numerator of the diviſor into the denominator of the dividend for a new denominator.*

Thus $\frac{a}{b}$ divided by $\frac{c}{d}$, gives $\frac{ad}{bc}$:

Alſo $\frac{5ax}{3c}$ divided by $\frac{6bc}{7d}$, gives $\frac{35adx}{18bc}$:

And $\frac{6a^2b}{5x}$ divided by $\frac{5ab^2}{3x}$ gives $\frac{18a^2bx}{25ab^2x}$.

---

juſt ſo many times likewiſe is the double, triple, quadruple, or any other aſſigned multiple of the former contained in the double, triple, quadruple, or other correſponding multiple of the latter. The Demonſtration of this Principle (though it may be thought too obvious to need one) may be thus: Let A and B repreſent any two propoſed quantities, and AC and BC their *equimultiples* (or, let AC and BC be the two quantities, and A and B their *like* parts): I ſay, then, that $\frac{AC}{BC} = \frac{A}{B}$: For the multiple of $\frac{AC}{BC}$ by BC is manifeſtly $= AC$; and $\frac{A}{B} \times BC$, the multiple of $\frac{A}{B}$ by the ſame BC is $= \frac{A \times BC}{B}$ (*by rule 2 in multiplication*) $= \frac{ACB}{B}$ (*vid. p.* 14 *and* 15) $= AC$: Therefore, ſeeing the equimultiples of the two propoſed quantities are the ſame, the quantities themſelves muſt neceſſarily be equal.

The ſecond Rule, given above, is nothing more than a bare application of the Principle here demonſtrated;

But in caſes like this laſt, where the two numerators, or the denominators, have factors common to both, the concluſion will become more neat by firſt caſting off ſuch common factors.

Thus, caſting away *ab* out of the two numerators, and *x* out of both the denominators, we have $\frac{6a}{5}$ to be divided by $\frac{5b}{3}$; whereof the quotient is $\frac{18a}{25b}$: In the ſame manner $\frac{12ac^3}{10bb} \div \frac{4acx}{5bd}$, or $\frac{3c^2}{2b} \div \frac{x}{d}$, gives $\frac{3c^2d}{2bx}$; and $\frac{6a\sqrt{xy}}{5c} \div \frac{7a^2\sqrt{xy}}{10bc}$, or $\frac{6}{1} \div \frac{7a}{2b}$ gives $\frac{12b}{7a}$.

When either the diviſor or the dividend is a *whole quantity* (inſtead of a fraction) it may be reduced to the form of a fraction, by writing an unit or 1, under it.

ſince, by caſting off the factors common to the dividend and diviſor (as directed in the rule) it is plain that we take *like* parts of thoſe quantities: therefore the quotient ariſing by dividing the one part by the other, will be the ſame as that ariſing by dividing one whole by the other.

As to Rule 3°, wherein it is aſſerted that $\frac{A}{B} \div \frac{C}{D} = \frac{AD}{BC}$, it is evident that AD and BC are equimultiples of the given quantities $\frac{A}{B}$ and $\frac{C}{D}$; becauſe $\frac{A}{B} \times BD$ is *(by Rule 2° in multiplication)* $= \frac{ABD}{B} = AD$, and $\frac{C}{D} \times BD = \frac{CBD}{D} = CB$: Whence it follows that the quotient of $\frac{A}{B}$ divided by $\frac{C}{D}$ will be the ſame with that of AD divided by BC; which, by Notation is $\frac{AD}{BC}$, as was to be ſhewn. The Grounds of the note ſubjoined to this Rule are theſe: By caſting away all factors common to

Thus $\frac{12ab}{5c}$ divided by $7d$ (or $\frac{7d}{1}$) gives $\frac{12ab}{35cd}$;

And $5a^2b$ (or $\frac{5a^2b}{1}$) divided by $\frac{9x^2}{3y}$ gives $\frac{15a^2by}{9xx}$.

4°. *Surd quantities under the ſame radical ſign, are divided by one another like rational quantities, only the quotient muſt ſtand under the given radical ſign.*

Thus, the quotient of $\sqrt{ab}$ by $\sqrt{b}$ is $\sqrt{a}$:

That of $\sqrt[3]{16xxy}$ by $\sqrt[3]{8xy}$ is $\sqrt[3]{2x}$:

That of $\sqrt{\frac{10abb}{3c}}$ by $\sqrt{\frac{5ab}{c}}$ is $\sqrt{\frac{10abbc}{15abc}}$, or $\sqrt{\frac{2b}{3}}$.

And that of $6ab\sqrt{10acxy}$ by $2a\sqrt{2cy}$ is $3b\sqrt{5ax}$.

5°. *Different powers, or roots of the ſame quantity are divided one by another, by ſubtracting the exponent of the diviſor from that of the dividend, and placing the remainder as an exponent to the quantity given.* But it muſt be obſerved, that the exponents here underſtood are thoſe defined in p. 5; where all roots are repreſented as fractional powers. It will likewiſe be proper to remark further, that, when the exponent of the diviſor is greater than that of the dividend, the quotient will have a negative exponent, or, which comes to the ſame thing, the reſult will be a fraction, whereof the numerator is an unit, and the denominator the ſame quantity with its exponent changed to an affirmative one.

Thus x^5 divided by x^2 gives x^3:

And $\overline{a+z}|^7$ divided by $\overline{a+z}|^3$ gives $\overline{a+z}|^4$:

Likewiſe $x^{\frac{1}{2}}$ divided by $x^{\frac{1}{4}}$ gives $x^{\frac{1}{4}}$:

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to the two numerators we take equal parts of the quantities; and by throwing off the factors common to both denominators, we take equimultiples of thoſe parts.

The two preceding Rules, being nothing more than the converſe of 4th and 5th Rules in multiplication, are demonſtrated in them: though perhaps the caſe, in Rule 5, where the exponent comes out negative, may ſtand in need of a more particular Explanation. According

Moreover, $\overline{c+y}|^{\frac{1}{2}}$ didided by $\overline{c+y}|^{\frac{1}{3}}$ gives $\overline{c+y}|^{\frac{1}{6}}$:

Laſtly, x^3 divided by x^5 gives x^{-2}, or $\frac{1}{x^2}$.

6°. *A compound quantity is divided by a ſimple one, by dividing every term thereof by the given diviſor.*

Thus, $3ab)\ 3abc + 12abx - 9aab\ (c + 4x - 3a$:

alſo, $-5ac)\,15a^2bc - 12acy^2 + 5ad^2(-3ab + \frac{12yy}{5} - \frac{dd}{c}$:

and ſo of others.

7°. *But if the diviſor, as well as the dividend, be a compound quantity, let the terms of both quantities be diſpoſed in order, according to the dimenſions of ſome letter in them, as ſhall be judged moſt expedient, ſo that thoſe terms may ſtand firſt wherein the higheſt power of that letter is involved, and thoſe next where the next higheſt power is involved, and ſo on: this being done, ſeek how many times the firſt term of the diviſor is contained in the firſt term of the dividend, which, when found, place in the quotient (as in diviſion in vulgar arithmetic) and then multiply the whole diviſor thereby, ſubtracting the product from the reſpective terms of the dividend; to the remainder bring down, with their proper ſigns, as many of the next following terms of the dividend as are requiſite for the next operation; ſeeking again how often the firſt term of the diviſor is contained in the firſt term of the remainder, which alſo write down in your quotient, and proceed as before, repeating the operation till all the terms of the dividend are exhauſted, and you have nothing remaining.*

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ing to the ſaid rule, the quotient of x^3 divided by x^5 was aſſerted to be x^{-2}, or $\frac{1}{x^2}$. Now that this is the true value is evident; becauſe 1 and x^2 being like parts of x^3 and x^5 (which ariſe by dividing by x^3) their quotient will conſequently be the ſame with that of the quantities themſelves.

Thus, if it were required to divide $a^3+5a^2x+5ax^2+x^3$ by $a+x$ (where the ſeveral terms are diſpoſed according to the dimenſions of the letter a) I firſt write down the diviſor and dividend, in the manner below, with a crooked line between them, as in the Diviſion of whole Numbers; then I ſay, how often is a contained in a^3, or what is the quotient of a^3 by a; the anſwer is a^2, which I write down in the quotient, and multiply the whole diviſor, $a+x$, thereby, and there ariſes a^3+a^2x; which ſubtracted from the two firſt terms of the dividend leaves $4a^2x$; to this remainder I bring down $+5ax^2$, the next term of the dividend, and then ſeek again how many times a is contained in $4a^2x$; the anſwer is $4ax$, which I alſo put down in the quotient, and by it multiply the whole diviſor, and there ariſes $4a^2x+4ax^2$, which ſubtracted from $4a^2x+5ax^2$ leaves ax^2, to which I bring down x^3, the laſt term of the dividend, and ſeek how many times a is contained in ax^2, which I find to be x^2; this I therefore alſo write down in the quotient, and by it multiply the whole diviſor; and then, having ſubtracted the product from ax^2+x^3, find there is nothing remains; whence I conclude, that the required quotient is truly expreſſed by $a^2+4ax+x^2$. See the operation.

$$\begin{array}{rcccccccl}
a+x) & a^3 & + & 5a^2x & + & 5ax^2 & + & x^3 & (a^2+4ax+x^2 \\
 & a^3 & + & a^2x & & & & & \\ \cline{2-4}
 & & & 4a^2x & + & 5ax^2 & & & \\
 & & & 4a^2x & + & 4ax^2 & & & \\ \cline{4-6}
 & & & & & ax^2 & + & x^3 & \\
 & & & & & ax^2 & + & x^3 & \\ \cline{6-8}
 & & & & & 0 & & 0 &
\end{array}$$

In the ſame manner, if it be propoſed to divide $a^5-5a^4x+10a^3x^2-10a^2x^3+5ax^4-x^5$ by $a^2-2ax+x^2$, the quotient will come out $a^3-3a^2x+3ax^2-x^3$, as will appear from the proceſs.

a^5-

$$\begin{array}{l}
\hspace{14em} -3a^2x+3ax^2-x^3 \\
a^2-2ax+x^2)\; a^5-5a^4x+10a^3x^2-10a^2x^3+5ax^4-x^5\;(a^3 \\
\hspace{6em} \underline{a^5-2a^4x+\quad a^3x^2 \qquad\qquad} \\
\hspace{6.5em} -3a^4x+\;9a^3x^2-10a^2x^3 \\
\hspace{6.5em} \underline{-3a^4x+\;6a^3x^2-\;\;3a^2x^3 \qquad\qquad} \\
\hspace{10em} +\;3a^3x^2-\;7a^2x^3+5ax^4 \\
\hspace{10em} \underline{+\;3a^3x^2-\;6a^2x^3+3ax^4 \qquad} \\
\hspace{14em} -\quad a^2x^3+2ax^4-x^5 \\
\hspace{14em} -\quad \underline{a^2x^3+2ax^4-x^5} \\
\hspace{17em} 0 \qquad 0 \qquad 0
\end{array}$$

So likewiſe, if a^5-x^5 be divided by $a-x$, the quotient will be $a^4+a^3x+a^2x^2+ax^3+x^4$; as by the work will appear.

$$\begin{array}{l}
a-x)\; a^5-x^5\;(a^4+a^3x+a^2x^2+ax^3+x^4 \\
\hspace{3em} \underline{a^5-a^4x \qquad} \\
\hspace{5em} a^4x-x^5 \\
\hspace{5em} \underline{a^4x-a^3x^2 \qquad} \\
\hspace{8em} a^3x^2-x^5 \\
\hspace{8em} \underline{a^3x^2-a^2x^3 \qquad} \\
\hspace{11em} a^2x^3-x^5 \\
\hspace{11em} \underline{a^2x^3-ax^4 \qquad} \\
\hspace{14em} ax^4-x^5 \\
\hspace{14em} \underline{ax^4-x^5} \\
\hspace{14.5em} 0 \qquad 0
\end{array}$$

Moreover, if it were required to divide $a^6-3a^4x^2+3a^2x^4-x^6$ by $a^3-3a^2x+3ax^2-x^3$, the proceſs will ſtand thus:

$$\begin{array}{l}
a^3-3a^2x+ \;\big)\; a^6-3a^4x^2+3a^2x^4-x^6\;(a^3+3a^2x+3ax^2+x^3 \\
\quad 3ax^2-x^3 \;\big)\; \underline{a^6-3a^5x+3a^4x^2-a^3x^3 \qquad} \\
\hspace{7em} +3a^5x-6a^4x^2+\;a^3x^3+3a^2x^4 \\
\hspace{7em} \underline{+3a^5x-9a^4x^2+9a^3x^3-3a^2x^4 \qquad} \\
\hspace{11em} +3a^4x^2-8a^3x^3+6a^2x^4-x^6 \\
\hspace{11em} \underline{+3a^4x^2-9a^3x^3+9a^2x^4-3ax^5 \qquad} \\
\hspace{15em} +\;a^3x^3-3a^2x^4+3ax^5-x^6 \\
\hspace{15em} \underline{+\;a^3x^3-3a^2x^4+3ax^5-x^6} \\
\hspace{16em} 0 \qquad 0 \qquad 0 \qquad 0
\end{array}$$

But it is to be obſerved, that it is not always that the work will terminate without leaving a remainder; and then this method is of little uſe; and in all theſe caſes, it will be moſt commodious to expreſs the quotient, in the manner of a fraction, by writing the diviſor under the dividend, with a line between them, as has been ſhewn in the method of notation.

It would be needleſs to offer any thing by way of demonſtration to the two laſt rules, the grounds thereof being already ſufficiently clear from what has been delivered in the laſt ſection, and the rules themſelves nothing more than the converſe of thoſe there demonſtrated. —I ſhall here ſhew the reaſon why, in diviſion (as well as multiplication) *like* ſigns produce $+$, and *unlike*, $-$. In order thereto it muſt firſt be obſerved, that according to the nature of diviſion, every quotient whatever multiplied by the given diviſor, ought to produce the given dividend; whence it is evident,

1. That $+a)\ +ab\ (\ +b$; becauſe $+a$ mult. by $+b$, gives $+ab$;
2. That $+a)\ -ab\ (\ -b$; becauſe $+a$ mult. by $-b$, gives $-ab$;
3. That $-a)\ +ab\ (\ -b$; becauſe $-a$ mult. by $-b$, gives $+ab$;
4. That $-a)\ -ab\ (\ +b$; becauſe $-a$ mult. by $+b$, gives $-ab$;

And theſe four, are all the caſes that can poſſibly happen in reſpect to the variation of the ſigns.

SECTION VI.

Of Involution.

INVOLUTION is the raiſing of powers from any propoſed root, and may be performed by the following Rules.

1°. *If the Quantity, or Root propoſed to be involved has no index, that is, if it be not itſelf a power or ſurd, the power thereof*

thereof will be repreſented by the ſame quantity under the given index, or exponent.

Thus, the fifth power of a is expreſſed a^5; and the ſeventh power of $a + z$ by $\overline{a + z}|^7$.

2°. *But if the quantity propoſed be itſelf a power, or ſurd, it will be involved by multiplying its exponent by the exponent of the propoſed power.*

Thus, the cube, or third power of a^2 is a^6; the fifth power of x^3 is x^{15}; the fourth power of $\overline{ax + yy}|^3$ is $\overline{ax+yy}|^{12}$; and the third power of $\overline{a - x}|^{\frac{1}{2}}$ is $\overline{a - x}|^{\frac{3}{2}}$.

3°. *A Quantity compoſed of ſeveral factors multiplied together, is involved by raiſing each factor to the power propoſed.*

Thus, the ſquare, or ſecond power of ab is a^2b^2; the cube, or third power of $2ab$ is $2^3a^3b^3$, or $8a^3b^3$; the fifth power of $3 \times \overline{aa - xx} \times \overline{a + b + c}$ is $243 \times \overline{aa - xx}|^5 \times \overline{a + b + c}|^5$; and the ſquare, or ſecond power of the radical quantity $a^{\frac{1}{2}} \times \overline{a + x}|^{\frac{1}{3}}$ is $a \times \overline{a + x}|^{\frac{2}{3}}$.

The firſt of the rules, here given, being mere notation, does not require, nor indeed admit of a demonſtration: The ſecond may be explained thus; let A^m be propoſed to be raiſed to the power whoſe exponent is n: then I ſay, that the power itſelf will be truly expreſſed by A^{mn}: For ſince (by notation) A^m is the ſame thing as $A \times A \times A \times A$, &c. continued to m factors, This, raiſed to the nth power, or multiplied n times, will, (by the general obſervations at p. 13) be equal to $A \times A \times A \times A \times A \times A$, &c. continued to n times m factors, that is, to mn factors; which, by notation, is A^{mn}. But the ſame thing may be otherwiſe demonſtrated, in a more general manner, by means of rule 5th

4° *A Fraction is involved, by raiſing both the numerator and the denominator to the power propoſed.*

Thus, the ſecond power of $\frac{a}{b}$ is $\frac{aa}{bb}$; the third power of $\frac{2ab}{3c}$ is $\frac{8a^3b^3}{27c^3}$; the fourth power of $\frac{2a^2b}{3c^2}$ is $\frac{16a^8b^4}{81c^8}$; the ſquare of $\frac{\sqrt{x}}{5}$, or $\frac{x^{\frac{1}{2}}}{5}$ is $\frac{x}{25}$; the cube of $\frac{x^{\frac{2}{3}}}{a^{\frac{2}{3}}}$ is $\frac{x^2}{a^2}$; and the ſixth power of $\frac{\overline{aa+xx}\rvert^{\frac{1}{3}}}{\overline{a-x}\rvert^{\frac{1}{2}}}$ is $\frac{\overline{aa+xx}\rvert^{2}}{\overline{a-x}\rvert^{3}}$.

When any quantity to be involved has the ſign — prefixed, the power itſelf, if the index is an odd number, muſt be expreſſed with the ſame negative ſign, but if an even number, with the contrary ſign, or +.

Thus the ſecond power of $-a$, or $-a \times -a$, is $+a^2$ (becauſe — into — produces +): alſo the cube

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in multiplication: For, ſince powers raiſed from the ſame root are multiplied by addition of their indices, it is evident that the ſquare of A^m (or $A^m \times A^m$) whether the exponent m be a whole number or a fraction, will be truly defined by A^{2m}: whence it likewiſe appears, that the cube of A^m (or $A^{2m} \times A^m$) will be defined by A^{3m}; and the fourth power of A^m (or $A^{3m} \times A^m$) by A^{4m}, *&c.*

The Reaſon of the third Rule is alſo grounded on the ſame *general obſervations*: For, in the firſt example, where the ſquare of ab is aſſerted to be a^2b^2, we know that ſquare to be $ab \times ab$ (by the definition of a ſquare), which quantity is *there* proved to be the ſame with $a \times b \times a \times b$, or $aa \times bb$. So likewiſe in the ſecond example, the cube of $2ab$, or $2ab \times 2ab \times 2ab$, will be $= 2 \times a \times b \times 2 \times a \times b \times 2 \times a \times b = 2 \times 2 \times 2 \times a \times a \times a \times b \times b \times b = 8 \times a^3 \times b^3 = 8a^3b^3$. And of $-a$, or $+a^2\times -a$ is $-a^3$ (becauſe + into — produces —) ſo likewiſe the fourth power of $-a$, or $-a^3\times -a$ is $+a^4$, and the fifth power, or $+a^4\times -a$, is $-a^5$, &c. &c. Hence it appears that all even Powers, whether raiſed from *poſitive* or *negative* Roots, will be *poſitive*.

5°. *Quantities compounded of ſeveral terms are involved by an actual multiplication of all their parts.*

Thus, if $a+b$ was propoſed to be involved to the ſixth power; by multiplying $a+b$ into itſelf, we ſhall firſt have $a^2+2ab+b^2$, which is the ſecond power of $a+b$; and this, again, multiplied by $a+b$, gives $a^3+3a^2b+3ab^2+b^3$, for the third power of $a+b$: whence, by proceeding on, in this manner, the ſixth power of $a+b$ will be found to come out $a^6+6a^5b+15a^4b^2+20a^3b^3+15a^2b^4+6ab^5+b^6$. See the operation.

$a+b$, the root or firſt power.

$$\begin{array}{l} \phantom{aa+{}}a+b \\ \hline aa+ab \\ +ab+b^2 \\ \hline a^2+2ab+b^2 \end{array}$$

the ſquare, or ſecond power.

$a+b$

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the caſe will be the ſame when radical quantities are concerned (as in the fourth example): For the ſquare of $a^{\frac{1}{2}}\times\overline{a+x})^{\frac{1}{3}}$, or $\overline{a^{\frac{1}{2}}\times\overline{a+x})^{\frac{1}{3}}}\times\overline{a^{\frac{1}{2}}\times\overline{a+x})^{\frac{1}{3}}}$ is $= a^{\frac{1}{2}}\times a^{\frac{1}{2}}\times\overline{a+x})^{\frac{1}{3}}\times\overline{a+x})^{\frac{1}{3}} = \overline{a^{\frac{1}{2}}\times a^{\frac{1}{2}}}\times\overline{\overline{a+x})^{\frac{1}{3}}\times\overline{a+x})^{\frac{1}{3}}}$: but $a^{\frac{1}{2}}\times a^{\frac{1}{2}}$ (by rule 5° in multiplication) is $=a^1=a$, and $\overline{a+x})^{\frac{1}{3}}\times\overline{a+x})^{\frac{1}{3}}=\overline{a+x})^{\frac{2}{3}}$; therefore our ſquare, or its equal product, is likewiſe expreſſed by $a\times\overline{a+x})^{\frac{2}{3}}$.

The 4th rule, or caſe, for the involution of fractions, is grounded on rule 3° in multiplication, and requires no other demonſtration than is there given.

$a^3 + 2a^2b + ab^2$
$ + a^2b + 2ab^2 + b^3$
$a^3 + 3a^2b + 3ab^2 + b^3$ the cube, or third power.
$a + b$
$a^4 + 3a^3b + 3a^2b^2 + ab^3$
$ + a^3b + 3a^2b^2 + 3ab^3 + b^4$
$a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$, the fourth power.
$a + b$
$a^5 + 4a^4b + 6a^3b^2 + 4a^2b^3 + ab^4$
$ + a^4b + 4a^3b^2 + 6a^2b^3 + 4ab^4 + b^5$
$a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$, the 5th power.
$a + b$
$a^6 + 5a^5b + 10a^4b^2 + 10a^3b^3 + 5a^2b^4 + ab^5$
$ + a^5b + 5a^4b^2 + 10a^3b^3 + 10a^2b^4 + 5ab^5 + b^6$
$a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$, the 6th, or required power of $a + b$.

So likewiſe, if it be required to involve or raiſe $a - b$ to the ſixth power, the Proceſs will ſtand thus:

$a - b$
$a - b$
$a^2 - ab$
$ - ab + b^2$
$a^2 - 2ab + b^2$, ſecond power.
$a - b$
$a^3 - 2a^2b + ab^2$
$ - a^2b + 2ab^2 - b^3$
$a^3 - 3a^2b + 3ab^2 - b^3$, third power.
$a - b$
$a^4 - 3a^3b + 3a^2b^2 - ab^3$
$ - a^3b + 3a^2b^2 - 3ab^3 + b^4$
$a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4$, fourth power.
$a - b$
$a^5 - 4a^4b + 6a^3b^2 - 4a^2b^3 + ab^4$
$ - a^4b + 4a^3b^2 - 6a^2b^3 + 4ab^4 - b^5$
$a^5 - 5a^4b + 10a^3b^2 - 10a^2b^3 + 5ab^4 - b^5$, fifth power.
$a - b$

$a^6 -$

$$\begin{array}{l} a^6 - 5a^5b + 10a^4b^2 - 10a^3b^3 + 5a^2b^4 - ab^5 \\ \quad - a^5b + 5a^4b^2 - 10a^3b^3 + 10a^2b^4 - 5ab^5 + b^6 \\ \hline a^6 - 6a^5b + 15a^4b^2 - 20a^3b^3 + 15a^2b^4 - 6ab^5 + b^6 \end{array}$$

the ſixth power of $a - b$; and ſo of any other.

But there is a Rule, or Theorem, given by Sir *Iſaac Newton*, (demonſtrated hereafter) whereby any power of a binomial $a + b$, or $a - b$, may be expreſſed in ſimple terms, without the trouble of thoſe tedious multiplications required in the preceding operations; which is thus:

Let n denote any number at pleaſure; then the nth power of $a + b$ will be $a^n + na^{n-1}b + \frac{n \cdot n-1}{1 \cdot 2} \cdot a^{n-2}b^2 + \frac{n \cdot n-1 \cdot n-2}{1 \cdot 2 \cdot 3} \cdot a^{n-3}b^3 + \frac{n \cdot n-1 \cdot n-2 \cdot n-3}{1 \cdot 2 \cdot 3 \cdot 4} \cdot a^{n-4}b^4 + \frac{n \cdot n-1 \cdot n-2 \cdot n-3 \cdot n-4}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} \cdot a^{n-5}b^5$, &c. And the nth power of $a - b$ will be expreſſed in the very ſame manner, only the ſigns of the ſecond, fourth, ſixth, &c. terms where the odd powers of b are involved, muſt be negative.

An example or two will ſhew the uſe of this general Theorem.

Firſt, then, let it be required to raiſe $a + b$ to the third power. Here n, the index of the propoſed power, being 3, the firſt term, a^n, of the general expreſſion, is equal to a^3; the ſecond $na^{n-1}b = 3a^2b$; the third $\frac{n \cdot n-1}{1 \cdot 2} \cdot a^{n-2}b^2 = 3ab^2$; the fourth $\frac{n \cdot n-1 \cdot n-2}{1 \cdot 2 \cdot 3} \cdot a^{n-3}b^3 = b^3$; and the fifth $\frac{n \cdot n-1 \cdot n-2 \cdot n-3}{1 \cdot 2 \cdot 3 \cdot 4} \cdot a^{n-4}b^4$, &c. $=$ nothing. Therefore the third power of $a+b$ is truly expreſſed by $a^3 + 3a^2b + 3ab^2 + b^3$.

Again, let it be required to raiſe $a + b$ to the ſixth power. In which caſe the index, n, being 6, we ſhall, by proceeding as in the laſt example, have $a^n = a^6$,

$na^{n-1}b = 6a^5b$, $\frac{n.\overline{n-1}}{1\,.\,2}.\,a^{n-2}b^2 = 15a^4b^2$, &c. and conſequently $\overline{a+b}|^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$; being the very ſame as was above determined by continual multiplication.

Laſtly, let it be propoſed to involve $cc + xy$ to the fourth power.

Here a muſt ſtand for cc, b for xy, and n for 4; then, by ſubſtituting theſe values, inſtead of a, b, and n, the general expreſſion will become $c^8 + 4c^6xy + 6c^4x^2y^2 + 4c^2x^3y^3 + x^4y^4$, the true value ſought.

From the preceding operations it may be obſerved, that the unciæ, or coefficients, increaſe till the indices of the two letters a and b become equal, or change values, and then return, or decreaſe again, according to the ſame order: therefore we need only find the coefficients of the firſt half of the terms in this manner; ſince, from theſe the reſt are given.

SECTION VII.

Of Evolution.

EVOLUTION, *or the Extraction of Roots, being directly the contrary to Involution, or raiſing of powers, is performed by converſe operations, viz. by the diviſion of indices, as Involution was by their multiplication.*

Thus the ſquare root of x^6, by dividing the exponent by 2, is found to be x^3; and the cube root of x^6, by dividing the exponent by 3, appears to be x^2: moreover, the biquadratic root of $\overline{a+x}|^8$ will be $\overline{a+x}|^2$; and the cube root of $\overline{aa+xx}|^2$ will be $\overline{aa+xx}|^{\frac{2}{3}}$.

In the ſame manner, if the quantity given be a fraction, or conſiſts of ſeveral factors multiplied together, its root will be extracted, by extracting the root of each particular factor.

Thus the ſquare root of a^2b^2 will be ab; that of $\frac{a^2b^2}{c^2}$ will be $\frac{ab}{c}$; and that of $\frac{81\times a^2\times\overline{aa+xx}|^{4}}{16\times\overline{a-x}|^{2}}$ will be $\frac{9a\times\overline{aa+xx}|^{2}}{4\times\overline{a-x}}$: Moreover, the ſquare root of $\overline{aa-xx}|^{\frac{1}{2}}$ will be $\overline{aa-xx}|^{\frac{1}{4}}$; its cube root $\overline{aa-xx}|^{\frac{1}{6}}$; and its biquadratic root, $\overline{aa-xx}|^{\frac{1}{8}}$; and ſo of others. All which being nothing more than the converſe of the operations in the preceding ſection, requires no other demonſtration than *what* is there given.

Evolution of compound quantities is performed by the following Rule.

Firſt, place the ſeveral Terms, whereof the given quantity is compoſed, in order, according to the dimenſions of ſome letter therein, as ſhall be judged moſt commodious; then let the root of the firſt term be found, and placed in the quotient; which term being ſubtracted, let the firſt term of the remainder be brought down, and divided by twice the firſt term of the quotient, or by three times its ſquare, or four times its cube, &c. *according as the root to be extracted is a ſquare, cubic, or biquadratic one,* &c. *and let the quantity thence ariſing be alſo wrote down in the quotient, and the whole be raiſed to the ſecond, third, or fourth,* &c. *power, according to the aforeſaid Caſes, reſpectively, and ſubtracted from the given quantity; and (if any thing remains) let the operation be repeated, by always dividing the firſt term of the remainder by the ſame diviſor, found as above.*

Suppoſe, for example, it were required to extract the ſquare root of the compound quantity $2ax + a^2 + x^2$: then having ranged the terms in order according to the dimenſions of the letter a, the given quantity will ſtand thus, $a^2 + 2ax + x^2$, and the root of its firſt term will be a; by the double of which I divide $2ax$, (the firſt of the remaining terms) and add $+ x$, the quantity thence ariſing to a (already found) and ſo have $a+x$ in the quotient; which being raiſed to the ſecond power, and ſubtracted from the given quantity, nothing remains; therefore $a + x$ is the ſquare root required. See the operation.

$$a^2 + 2ax + x^2 \;(a + x$$
$$2a)\;2ax$$
$$\underline{a^2 + 2ax + x^2}, \text{ ſecond power of } a + x.$$
$$0 \quad 0 \quad 0$$

In like manner, if the quantity $a^4 - 2a^3x + 3a^2x^2 - 2ax^3 + x^4$ be propoſed to extract the ſquare root thereof; the anſwer will come out $a^2 - ax + x^2$, as appears by the proceſs.

$$a^4 - 2a^3x + 3a^2x^2 - 2ax^3 + x^4 \;(a^2 - ax + x^2$$
$$2a^2)\;-2a^3x$$
$$\underline{a^4 - 2a^3x + a^2x^2}, \text{ ſecond power of } a^2 - ax.$$
$$2a^2)\quad 2a^2x^2, \text{ firſt term of the remainder.}$$
$$\underline{a^4 - 2a^3x + 3a^2x^2 - 2ax^3 + x^4}, \text{ ſquare of } a^2 - ax + x^2.$$
$$0 \quad 0 \quad 0 \quad 0 \quad 0$$

Again, let it be required to extract the cube root of $a^3 - 6a^2x + 12ax^2 - 8x^3$, and the work will ſtand thus:

$$a^3 - 6a^2x + 12ax^2 - 8x^3 \;(a - 2x$$
$$3a^2)\;-6a^2x$$
$$\underline{a^3 - 6a^2x + 12ax^2 - 8x^3}, \text{ cube of } a - 2x.$$
$$0 \quad 0 \quad 0 \quad 0$$

Laſtly, let it be required to extract the biquadratic root of $16x^4 - 96x^3y + 216x^2y^2 - 216xy^3 + 81y^4$, and the proceſs will ſtand as follows:

$$16x^4 - 96x^3y + 216x^2y^2 - 216xy^3 + 81y^4 \;(2x - 3y$$
$$\underline{32x^3)\;-96x^3y}$$
$$\underline{16x^4 - 96x^3y + 216x^2y^2 - 216xy^3 + 81y^4}.$$
$$0 \quad 0 \quad 0 \quad 0 \quad 0$$

And, in the ſame manner the root may be determined in any other caſe, where it is poſſible to be extracted; but if that cannot be done, or, after all, there is a remainder, then the root is to be expreſſed in the manner of a ſurd, according to what has been already ſhewn. As to the truth of the preceding Rule, it is too obvious to need a formal demonſtration, every operation being a proof of itſelf. I ſhall only add here, that there are other Rules beſides *that*, for extracting the roots of compound quantities; which, ſometimes, bring out the concluſions rather more expeditiouſly; but as theſe are confined to particular caſes, and would take up a great deal of room to explain in a manner ſufficiently clear and intelligible, it ſeemed more eligible to lay down the whole in one eaſy general method, than to diſcourage and retard the Learner by a multiplicity of Rules.—However, as the extraction of the ſquare root is much more neceſſary and uſeful than the reſt, I ſhall here put down one ſingle example thereof, wrought according to the common method of extracting the ſquare root, in numbers; which I ſuppoſe the reader to be acquainted with, and which he will find more expeditious than the general Rule explained above.

$$\begin{array}{rl}
\textit{Examp.} & a^4+4a^3x+6a^2x^2+4ax^3+x^4\ (a^2+2ax+x^2 \\
& a^4 \\ \hline
2a^2+2ax) & +4a^3x+6a^2x^2 \\
& +4a^3x+4a^2x^2 \\ \hline
2a^2+4ax+x^2) & +2a^2x^2+4ax^3+x^4 \\
& +2a^2x^2+4ax^3+x^4 \\ \hline
& 0 \quad 0 \quad 0
\end{array}$$

SECTION VIII.

Of the Reduction of fractional and radical Quantities.

THE Reduction of fractional and radical quantities is of uſe in changing an expreſſion to the moſt ſimple and commodious form it is capable of; and that, either by bringing it to its leaſt terms, or all the members thereof (if it be compounded) to the ſame denomination.

A Fraction is reduced to its leaſt terms, by dividing both the numerator and denominator by the greateſt common diviſor.

Thus, $\frac{ab}{bc}$, by dividing by b, is reduced to $\frac{a}{c}$:

And $\frac{2abc}{abb}$, by dividing by ab, is reduced to $\frac{2c}{b}$:

Moreover, $\frac{20abd}{5ab}$ will be reduced to $\frac{4d}{1}$, or $4d$:

And $\frac{12ax^2\sqrt{xy}}{72a^2x^2\sqrt{xy}}$ will be reduced to $\frac{1}{6a}$.

Thus alſo, $\frac{12aa-2ab}{4a^2}$, by dividing every term of the numerator and denominator by $2a$, is reduced to $\frac{6a-b}{2a}$:

And $\frac{8a^3x-12a^2x^2+6ax^3}{6a^2x+4ax^2}$, by dividing every term by $2ax$, is reduced to $\frac{4a^2-6ax+3x^2}{3a+2x}$:

Laſtly, $\frac{a^3+3a^2b+3ab^2+b^3}{a^2+3ab+2b^2}$, by dividing both the numerator and denominator by the compound diviſor $a+b$, is reduced to $\frac{a^2+2abd+bb}{a+2b}$.

But the compound diviſors whereby a Fraction can, ſometimes, be reduced to lower terms, are not ſo eaſily diſcovered as its ſimple ones; for which reaſon, it may not be improper to lay down a Rule for finding ſuch diviſors.

Firſt, divide both the numerator and denominator by their greateſt ſimple diviſors, and then the quotients one by the other (as is taught in Caſe 7. Section 5.) *always obſerving to make that the diviſor which is of the leaſt dimenſions; and if any thing remains, divide it by its greateſt ſimple diviſor, and then divide the laſt compound di-*

viſor by the quantity thence ariſing; and if any thing yet remains, divide it likewiſe by its greateſt ſimple diviſor, and the laſt compound diviſor by the quantity thence ariſing; proceed on in this manner till nothing remains; ſo ſhall the laſt diviſor exactly divide both the numerator and denominator, without leaving any remainder.

Note, If, after you have divided any remainder by its ſimple diviſor, you can diſcover a compound one which will likewiſe meaſure the ſame, and is prime to the diviſor from whence that remainder aroſe, it will be convenient to divide, alſo, thereby. And, if in any caſe it ſhould happen that the firſt term of the diviſor does not exactly meaſure that of the dividend, the whole dividend may be multiplied by any quantity, as ſhall be neceſſary to make the operation ſucceed.

Ex. 1. Let it be required to reduce the Fraction $\frac{5a^5 + 10a^4b + 5a^3b^2}{a^3b + 2a^2b^2 + 2ab^3 + b^4}$ to its loweſt terms, or to find the greateſt common meaſure of its numerator and denominator. Here, dividing firſt by the greateſt ſimple diviſors, $5a^3$ and b, we have $a^2 + 2ab + b^2$, and $a^3 + 2a^2b + 2ab^2 + b^3$; and if the latter of theſe be divided by the former, the work will ſtand thus:

$$\begin{array}{l} a^2 + 2ab + b^2)\ a^3 + 2a^2b + 2ab^2 + b^3\ (a \\ \qquad\qquad\qquad\ \underline{a^3 + 2a^2b + \ ab^2} \end{array}$$

where the remainder is $+ ab^2 + b^3$; which being divided by b^2, its greateſt ſimple diviſor, gives $a + b$; by *this* divide $a^2 + 2ab + b^2$, and the quotient will come out $a + b$, exactly; therefore the laſt diviſor, $a + b$, will exactly meaſure both quantities, as may be proved thus:

$$\begin{array}{l} a + b\ )\ 5a^5 + 10a^4b + 5a^3b^2\ (5a^4 + 5a^3b \\ \qquad\quad\ \underline{5a^5 + \ 5a^4b\qquad\qquad\ } \\ \qquad\qquad\qquad\ 5a^4b + 5a^3b^2 \\ \qquad\qquad\qquad\ \underline{5a^4b + 5a^3b^2} \\ \qquad\qquad\qquad\quad\ 0 \qquad\ 0 \end{array}$$

$a + b)$

$$\begin{array}{rl}
a+b) & a^3b + 2a^2b^2 + 2ab^3 + b^4\ (a^2b + ab^2 + b^3 \\
& \underline{a^3b + a^2b^2} \\
& \qquad a^2b^2 + 2ab^3 \\
& \qquad \underline{a^2b^2 + ab^3} \\
& \qquad\qquad ab^3 + b^4 \\
& \qquad\qquad \underline{ab^3 + b^4} \\
& \qquad\qquad 0 \quad 0
\end{array}$$

In both which caſes nothing remains; therefore the fraction given will be reduced to $\frac{5a^4 + 5a^3b}{a^2b + ab^2 + b^3}$.

Ex. 2. Let it be propoſed to reduce the fraction $\frac{a^4 - x^4}{a^3 - a^2x - ax^2 + x^3}$ to its loweſt terms: then the work will ſtand as follows;

$$\begin{array}{rl}
a^3 - a^2x - ax^2 + x^3) & a^4 + 0 + 0 + 0 - x^4\ (a + x \\
& \underline{a^4 - a^3x - a^2x^2 + ax^3} \\
& \qquad a^3x + a^2x^2 - ax^3 - x^4 \\
& \qquad \underline{a^3x - a^2x^2 - ax^3 + x^4} \\
& \qquad\quad + 2a^2x^2 + 0 - 2x^4
\end{array}$$

$$\begin{array}{rl}
a^2 + 0 - x^2) & a^3 - a^2x - ax^2 + x^3\ (a - x \\
& \underline{a^3 - 0 - ax^2} \\
& - a^2x + 0 + x^3 \\
& \underline{- a^2x + 0 + x^3} \\
& \quad 0 \quad 0 \quad 0
\end{array}$$

From whence it appears that $a^2 + 0 - x^2$, or $a^2 - x^2$ will meaſure both $a^4 - x^4$ and $a^3 - a^2x - ax^2 + x^3$; and, by dividing thereby, the fraction propoſed is reduced to $\frac{a^2 + x^2}{a - x}$.

Theſe Operations are founded on this Principle, *That whatever quantity meaſures the whole, and one part of another, muſt do the like by the remaining part.* For, that quantity (whatever it is) which meaſures both the diviſor and dividend, in the firſt example, muſt evidently meaſure $a^3 + 2a^2b + ab^2$ (being a multiple of the

Example 3. In the ſame manner the fraction $\frac{x^4 - 3ax^3 - 8a^2x^2 + 18a^3x - 8a^4}{x^3 - ax^2 - 8a^2x + 6a^3}$ will be reduced to $\frac{x^2 - 5ax + 4a^2}{x - 3a}$. See the proceſs.

$$x^3-ax^2-8a^2x+6a^3)\,x^4-3ax^3-8a^2x^2+18a^3x-8a^4\,(x-2a$$
$$x^4-ax^3-8a^2x^2+6a^3x$$
$$\overline{-2ax^3+0+12a^3x-8a^4}$$
$$-2ax^3+2a^2x^2+16a^3x-12a^4$$
$$\text{remainder}\ \overline{-2a^2x^2-4a^3x+4a^4};$$

which divided by $-2a^2$, gives $x^2 + 2ax - 2a^2$ for the next diviſor.

$$x^2+2ax-2a^2)\,x^3-ax^2-8a^2x+6a^3\,(x-3a$$
$$x^3+2ax^2-2a^2x$$
$$\overline{-3ax^2-6a^2x+6a^3}$$
$$-3ax^2-6a^2x+6a^3$$
$$\overline{0\quad 0\quad 0}$$

$$x^2+2ax-2a^2)\,x^4-3ax^3-8a^2x^2+18a^3x-8a^4\,(x^2-5ax+4a^2$$
$$x^4+2ax^3-2a^2x^2$$
$$\overline{-5ax^3-6a^2x^2+18a^3x}$$
$$-5ax^3-10a^2x^2+10a^3x$$
$$\overline{+4a^2x^2+8a^3x-8a^4}$$
$$+4a^2x^2+8a^3x-8a^4$$
$$\overline{0\quad 0\quad 0}$$

Now if, by proceeding in this manner, no compound diviſor can be found, that is, if the laſt remainder be only a ſimple quantity, we may conclude the caſe propoſed does not admit of *any*, but is already in its loweſt terms.

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the former): whence, by the Principle above quoted, the ſame quantity, as it meaſures the whole dividend, muſt alſo meaſure the remaining part of it, $ab^2 + b^3$: but the diviſor, we are in queſt of, being a compound one, we may caſt off the ſimple diviſor b^2, as not for our purpoſe; whence $a + b$ appears to be the only compound diviſor the caſe admits of: which, therefore, muſt be the common meaſure required, if the example propoſed admits of any ſuch.

terms. Thus, for inſtance, if the fraction propoſed were to be $\frac{a^3 + 2a^2x + 3ax^2 + 4x^3}{a^2 + ax + x^2}$; it is plain, by inſpection, that it is not reducible by any ſimple diviſor; but to know whether it may not, by a compound one, I proceed as above, and find the laſt remainder to be the ſimple quantity $7xx$: whence I conclude that the fraction is already in its loweſt terms.

Another obſervation may be here made, in relation to fractions that have in them more than two different letters. When one of the letters riſes only to a ſingle dimenſion, either in the numerator or in the denominator, it will be beſt to divide the ſaid numerator or denominator (whichever it is) into two parts, ſo that the ſaid letter may be found in every term of the one part, and be totally excluded out of the other; this being done, let the greateſt common diviſor of theſe two parts be found; which will, evidently, be a diviſor to the *whole*, and by which the diviſion of the other quantity is to be tried; as in the following example, where the fraction given is $\frac{x^3 + ax^2 + bx^2 - 2a^2x + bax - 2ba^2}{xx - bx + 2ax - 2ab}$. Here the denominator being the leaſt compounded, and b riſing therein to a ſingle dimenſion only, I divide the *ſame* into the parts $x^2 + 2ax$, and $-\overline{bx - 2ab}$; which, by inſpection, appear to be equal to $\overline{x + 2a} \times x$, and $\overline{x + 2a} \times -b$. Therefore $x + 2a$ is a diviſor to both the parts, and likewiſe to the whole, expreſſed by $\overline{x + 2a} \times \overline{x - b}$; ſo that one of theſe two factors, if the fraction given can be reduced to lower terms, muſt alſo meaſure the numerator: but the former will be found to ſucceed, the quotient coming out $x^2 - ax + bx - ab$, *exactly*: whence the fraction itſelf is reduced to $\frac{x^2 - ax + bx - ab}{x - b}$; which is not reducible farther, by $x - b$, ſince the diviſion does not terminate without a remainder, as upon trial will be found.

Having inſiſted largely on the reduction of fractions to their leaſt terms, we now come to conſider their reduction to the ſame denominator.

Fractions are reduced to the ſame denominator by multiplying the numerator of each into all the denominators, except its own, for a new correſponding numerator, and all the denominators continually together, for a common denominator.

Thus, $\frac{a}{b}$ and $\frac{c}{d}$ will be reduced to $\frac{ad}{bd}$ and $\frac{bc}{bd}$;

$\frac{a}{b}$, $\frac{c}{d}$, and $\frac{e}{f}$, to $\frac{adf}{bdf}$, $\frac{cbf}{bdf}$, and $\frac{bde}{bdf}$;

and $\frac{2ax}{cd}$, and $\frac{5bx}{3a}$, to $\frac{6a^2x}{3acd}$, and $\frac{5bxcd}{3acd}$; and

ſo of others.

But when the denominators have a common diviſor, the operation will be more ſimple, and the concluſion neater, if, inſtead of multiplying the terms of each fraction by the denominator of the other, you only multiply by that part which ariſes by dividing by the common diviſor. As, if there were propoſed the fractions $\frac{b^2}{ad}$ and $\frac{ab}{cd}$; then, the denominators having the factor d common to both, I multiply by the remaining factors a and c; whence the two fractions will be reduced to $\frac{bbc}{acd}$ and $\frac{aab}{acd}$ (where d remains as before, nothing having been done therewith). By the ſame method $\frac{3xy^3}{5abc}$ and $\frac{7bx^3}{4abd}$ are reduced to $\frac{12dxy^3}{20abcd}$ and $\frac{35bcx^3}{20abcd}$; and $\frac{6a\sqrt{ax}}{5bc}$ and $\frac{7c\sqrt{aa+xx}}{3ab}$, to $\frac{18a^2\sqrt{ax}}{15abc}$ and $\frac{35c^2\sqrt{aa+xx}}{15abc}$.

But, as has been before hinted, the principal uſe of this ſort of reduction is to transform compound quantities to the moſt commodious forms of expreſſion;

which, for the general part, are more eaſily managed, (whether they are to be added, ſubtracted, multiplied, or divided) when all their members are brought to the ſame denomination.

Thus the compound quantity $\frac{a}{b} + \frac{c}{d}$ will be transformed to $\frac{ad}{bd} + \frac{bc}{bd}$ or to $\frac{ad + bc}{bd}$; for it is evident, that the quotient which ariſes by dividing the whole, is equal to the quotients of all the parts, by the ſame diviſor.

In the ſame manner will $\frac{a}{b} - \frac{c}{d}$ be $= \frac{ad - bc}{bd}$;

and $\frac{2xy}{5aa} + \frac{3x}{b} - \frac{c}{d} = \frac{2xybd + 15aaxd - 5aabc}{5aabd}$;

alſo $\frac{2dx}{a} + b$, or $\frac{2dx}{a} + \frac{b}{1}$ will be $= \frac{2dx + ab}{a}$;

and $\frac{2ab}{a-b} + a = \frac{2ab + aa - ab}{a-b} = \frac{ab + aa}{a-b}$:

So likewiſe, by reduction, $\frac{a^2}{a-x} + \frac{a^2}{a+x} - 2a$

will be $= \frac{a^2 \times \overline{a+x} + a^2 \times \overline{a-x} - 2a \times \overline{a+x} \times \overline{a-x}}{\overline{a-x} \times \overline{a+x}}$

$= \frac{2ax^2}{a^2 - x^2}$; and $\frac{5\sqrt{xy}}{a} + \frac{10a - 5x}{\sqrt{xy} + a} =$

$\frac{5xy + 5a\sqrt{xy} + 10a^2 - 5ax}{a\sqrt{xy} + a^2}$; and ſo in other caſes.

The reaſon of the two kinds of reduction hitherto explained, is grounded on this obvious principle, that the equimultiples, or like parts of quantities, are in the ſame ratio to each other, as the quantities themſelves, or, that the quotient which ariſes by dividing one quantity by another, is the ſame as ariſes by dividing any part or multiple of the former, by the like part or multiple of the latter:

Beſides theſe, there are yet two other ſorts of reduction which Authors have treated of under the head of fractions; which are, *the reducing of a whole quantity to an equivalent fraction of a given denomination*, and *a compound fraction to a ſimple one of the ſame value.* Neither of theſe, indeed, are of any great uſe in the ſolution of problems, however it might be improper to leave them intirely untouched.

1°. *A whole quantity is reduced to an equivalent fraction by multiplying it by the given denominator, and writing the multiplier underneath the product, with a line between them.*

Thus the quantity *a*, reduced to the denominator *b*, will be $\frac{ab}{b}$, and the quantity $c + d$, to the denominator $a + b$, will be $\frac{\overline{a + b} \times \overline{c + d}}{a + b}$ or $\frac{ac + bc + ad + bd}{a + b}$.

2°. *A compound fraction is reduced to a ſimple one of the ſame value, by multiplying the numerators together for a new numerator, and the denominators together for a new denominator.*

But by a compound fraction here, we are not to un-

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latter: for in reducing to the loweſt terms, it is plain that, inſtead of the whole numerator and denominator, we only take that part of each which is defined by the greateſt common meaſure; whereas, in reduction to the ſame denominator, we, on the contrary, make uſe of equimultiples of thoſe quantities; ſince, in multiplying any numerator into all the denominators, except its own, we multiply *it* by the very ſame quantities by which its denominator is multiplied.

The Rule, for reducing a compound fraction to a ſimple one, may be explained thus. It is plain that the part of $\frac{c}{d}$ defined by $\frac{1}{b}$; which ariſes by dividing by *b*, will be equal to $\frac{c}{bd}$ (the diviſor here derſtand one conſiſting of ſeveral terms, connected together by the ſigns + and — (which is the general definition of a compound quantity) but ſuch an one as expreſſes a given part of ſome other fraction.

Thus $\frac{2}{3}$ of $\frac{3a}{5}$ will be equal to $\frac{6a}{15}$ and the $\frac{a}{b}$ part of $\frac{c}{d}$ will be $= \frac{ac}{bd}$.

Of the Reduction of Radical Quantities.

The Reduction of ſurd quantities, like that of fractions, may be either to the leaſt terms, or to the ſame denomination.

A radical quantity is reduced to its leaſt terms, by reſolving it into two factors, and extracting the root of that which is rational.

Thus, $\sqrt{28}$ is reduced to $\sqrt{4} \times \sqrt{7}$; which, by extracting the ſquare root of 4, becomes $2\sqrt{7}$: alſo $\sqrt{a^2b}$ is reduced to $\sqrt{a^2} \times \sqrt{b}$; which, by extracting the root of a^2, becomes $a\sqrt{b}$: likewiſe $\sqrt[3]{a^3b^4c^2}$, or $\overline{a^3b^4c^2})^{\frac{1}{3}}$ is reduced to $\sqrt[3]{a^3b^3} \times \sqrt[3]{bc^2}$, or $ab\sqrt[3]{bc^2}$: moreover $\sqrt{\frac{4a^3x - 4a^2x^2}{bc^2}}$ is reduced to $\sqrt{\frac{4a^2}{c^2}} \times \sqrt{\frac{ax-x^2}{b}}$ or $\frac{2a}{c} \times \sqrt{\frac{ax-x^2}{b}}$; and $\sqrt[4]{\frac{16a^5x^7+16a^6x^6}{81b^4c^2-162b^5x}}$

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being b times as great); therefore the part of $\frac{c}{d}$ defined by $\frac{a}{b}$, being a times as great as that defined by $\frac{1}{b}$, muſt be truly expreſſed by $\frac{c}{bd} \times a$, or its equal $\frac{ac}{bd}$; *as was to be ſhewn.*

is reduced to $\sqrt[4]{\frac{16a^4x^4}{81b^4}} \times \sqrt[4]{\frac{ax^3 + a^2x^2}{c^2 - 2bx}}$ or $\frac{2ax}{3b} \times \sqrt[4]{\frac{ax^3 + a^2x^2}{c^2 - 2bx}}$; and ſo of any other: all which is evident from caſe 4 of multiplication, and caſe 3 of involution. But it is to be obſerved, that, in reſolving any expreſſion in this manner, the factor out of which the root is to be extracted, is always to be taken the greateſt the caſe will admit of. It alſo may be proper to take notice, that this kind of reduction is chiefly uſeful in the addition and ſubtraction of ſurd quantities, and in uniting the terms of compound expreſſions that are commenſurable to each other, where the irrational part, or factor, after reduction, is the ſame in each term.

Thus $\sqrt{18} + \sqrt{32}$ is reduced to $3\sqrt{2} + 4\sqrt{2}$, or $7\sqrt{2}$; and $\sqrt{8a^2} + \sqrt{50a^2} - \sqrt{72a^2}$ is reduced to $2a\sqrt{2} + 5a\sqrt{2} - 6a\sqrt{2} = a\sqrt{2}$. Moreover, by reduction, $\sqrt{\frac{12a^2x}{5}} + \sqrt{\frac{15a^2x}{4}}$ becomes $= \sqrt{\frac{48a^2x}{20}} + \sqrt{\frac{75a^2x}{20}} = 4a\sqrt{\frac{3x}{20}} + 5a\sqrt{\frac{3x}{20}} = 9a\sqrt{\frac{3x}{20}}$. And $3a\sqrt{4a^2x^2 + 8x^4} + 3x\sqrt{9a^4 + 18a^2x^2}$ becomes $6ax\sqrt{a^2 + 2x^2} + 9ax\sqrt{a^2 + 2x^2} = 15ax\sqrt{a^2 + 2x^2}$.

Surd quantities, under different radical ſigns, are reduced to the ſame radical ſign, by reducing their indices to the leaſt common denominator.

Thus $a^{\frac{1}{2}}$ and $a^{\frac{1}{3}}$, reduced to the ſame radical ſign, will become $a^{\frac{3}{6}}$ and $a^{\frac{2}{6}}$ (for the indices are here $\frac{1}{2}$ and $\frac{1}{3}$, and theſe are equivalent to $\frac{3}{6}$ and $\frac{2}{6}$, where both have the ſame denominator). In the ſame manner $\overline{2}|^{\frac{1}{2}}$ and $\overline{3}|^{\frac{1}{3}}$ will become $\overline{2}|^{\frac{3}{6}}$ and $\overline{3}|^{\frac{2}{6}}$, or $\overline{8}|^{\frac{1}{6}}$ and $\overline{9}|^{\frac{1}{6}}$. And, *univerſally*, $A^{\frac{n}{m}}$ and $B^{\frac{p}{q}}$, will, when their exponents are reduced to the ſame denomination, become $\overline{A}|^{\frac{nq}{mq}}$ and $\overline{B}|^{\frac{mp}{mq}}$.

The principal uſe of this ſort of reduction, is, when quantities under different radical ſigns are to be multiplied or divided by each other.

That the reduction of a radical quantity to another of a different denomination, by an equal multiplication of the terms of its exponent, makes no alteration in the value of the quantity, may be thus demonſtrated.

Let $A^{\frac{m}{n}}$ be any quantity of this kind; then, the terms of its exponent being equally multiplied by any number r, I ſay, the quantity $A^{\frac{mr}{nr}}$, hence ariſing, is equal to the given one $A^{\frac{m}{n}}$.

For, if x be aſſumed $= A^{\frac{1}{nr}}$, or, which is the ſame, if the value of x be ſuch, that $x^{nr} = A$; then, the nth root of x^{nr} being x^{r} *(by caſe 2 of ſection 6)* and the nth root of A being $A^{\frac{1}{n}}$ *(by notation)*, theſe two quantities x^{r} and $A^{\frac{1}{n}}$ muſt, likewiſe, be equal to each other: and, if they be both raiſed to the mth power, the equality will ſtill continue; but the mth power of the former (x^{r}) is $= x^{mr}$ *(by caſe 2 of involution)*; and the mth power of the latter $(A^{\frac{1}{r}})$ is $A^{\frac{m}{n}}$ *(by notation)*; therefore x^{mr} is $= A^{\frac{m}{n}}$. But, x being $= A^{\frac{1}{nr}}$, we have $x^{mr} = A^{\frac{mr}{nr}}$, *by notation*; and conſequently $A^{\frac{mr}{nr}} = A^{\frac{m}{n}}$; *which was to be proved.*

Thus, $\sqrt{5}$ multiplied by $\sqrt[3]{10}$, or $\overline{5}|^{\frac{1}{2}}$ into $\overline{10}|^{\frac{1}{3}}$, will give $\overline{125}|^{\frac{1}{6}} \times \overline{100}|^{\frac{1}{6}}$ or $\overline{12500}|^{\frac{1}{6}}$: alſo $\sqrt{ax}$ into $\sqrt[3]{a^2x}$, or $\overline{ax}|^{\frac{1}{2}}$ into $\overline{a^2x}|^{\frac{1}{3}}$ will give $\overline{a^3x^3}|^{\frac{1}{6}} \times \overline{a^4x^2}|^{\frac{1}{6}}$ or $\overline{a^7x^5}|^{\frac{1}{6}}$: and $\sqrt{ax}$ divided by $\sqrt[3]{a^2x}$ will give $\frac{\overline{a^3x^3}|^{\frac{1}{6}}}{\overline{a^4x^2}|^{\frac{1}{6}}}$, or $\overline{\frac{x}{a}}|^{\frac{1}{6}}$. Laſtly, $2x$ multiplied into $\sqrt{3ax}$, will give $\sqrt{4x^2} \times \sqrt{3ax}$, or $\sqrt{12ax^3}$,

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SECTION IX.

Of Equations.

AN Equation is, when two equal quantities, differently expreſſed, are compared together, by means of the ſign = placed between them.

Thus, $8-2=6$ is an equation, expreſſing the equality of the quantities $8-2$, and 6: and $x=a+b$ is an equation, ſhewing that the quantity repreſented by x is equal to the ſum of the two quantities repreſented by a and b.

Equations are the means whereby we come at ſuch concluſions as anſwer the conditions of a problem; wherein, from the quantities given, the unknown ones are determined; and this is called the reſolution, or reduction of equations.

Reduction of ſingle Equations.

Single equations are ſuch as contain only one unknown quantity; which, before that quantity can be diſcovered, muſt be ſo ordered and transformed, by the addition, ſubtraction, multiplication, or diviſion, *&c.* of equal quantities, that a juſt equality between the two parts thereof may be *ſtill* preſerved, and that there may reſult, at laſt, an equation, wherein the unknown quantity ſtands alone on one ſide, and all the known ones on the other. But, though this method of ordering an equation is grounded upon ſelf-evident principles, yet the operations are ſometimes a little difficult to manage in the beſt manner; for which reaſon the following Rules are ſubjoined.

1°. *Any Term of an equation, may be tranſpoſed to the contrary ſide, if its ſign be changed* *.

Thus, if $x + 6 = 16$; then will $x = 16 - 6$, that is, $x = 10$:

And, if $x - 4 = 8$; then will $x = 8 + 4$, or $x = 12$:

Alſo, if $3x = 2x + 24$; then will $3x - 2x = 24$, that is $x = 24$:

Again, if $5x - 8 = 3x + 20$; then will $5x - 3x = 20 + 8$, or $2x = 28$:

Laſtly, if $ax + bx - c + d - ex = f - g + hx - kx$; then, by tranſpoſition, $ax + bx - ex - hx + kx = f - g + c - d$; where all the terms affected by x (the unknown quantity) ſtand, now, on the ſame ſide of the equation.

2°. *If there is any quantity by which all the terms of the equation are multiplied, let them* all *be divided by that quantity; but if all of them be divided by any quantity, let the common diviſor be caſt away.*

Thus, the equation $ax = ab$ is reduced to $x = b$; alſo, $10x = 70$ (or $10 \times x = 10 \times 7$) is reduced to $x = 7$; and $x^3 = ax^2 + bx^2$, is reduced to $x = a + b$:

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[*] The reaſon of this rule is extremely evident; ſince tranſpoſing of a quantity thus, is nothing more than ſubtracting or adding *it* on both ſides of the equation, according as the ſign thereof is poſitive or negative. Thus, in the equation $x + 6 = 16$ (which by tranſpoſition becomes $x = 16 - 6 = 10$) the number 6 is ſubtracted from both ſides; and, in the equation $x - 4 = 8$ (which by tranſpoſition becomes $x = 8 + 4 = 12$) the number 4 is added on both ſides.

Moreover (*by the latter part of the Rule*) $\frac{x}{a} = \frac{b}{a}$ is reduced to $x = b$; and $\frac{ax^3}{c} = \frac{abx^2 - acx^2}{c}$, to $ax^3 = abx^2 - acx^2$; which, if the whole be divided by ax^2, will be farther reduced to $x = b - c$.

3°. *If there are irreducible fractions, let the whole equation be multiplied by the product of all their denominators, or, which is the ſame, let the numerator of every term in the equation be multiplied by all the denominators, except its own, ſuppoſing ſuch terms (if any there be) that ſtand without a denominator, to have an unit ſubſcribed.*

Thus, the equation $x + \frac{x}{2} + \frac{x}{3} = 11$, is reduced to $6x + 3x + 2x = 66$; and $x + \frac{x+2}{5} = 12 + \frac{x-3}{8}$, to $40x + 8x + 16 = 480 + 5x - 15$: ſo likewiſe $a - \frac{x}{a} = \frac{x+b}{c}$, is reduced to $a^2c - cx = ax + ab$; and $\frac{ax}{a+x} + a = \frac{cx}{b}$, to $abx + a^2b + abx = acx + cx^2$.

4° *If, in your equation, there is an irreducible ſurd, wherein the unknown quantity enters, let all the other terms be tranſpoſed to the contrary ſide* (by rule 1.); *and then, if both ſides be involved to the power denominated by the ſurd, an equation will ariſe free from radical quantities; unleſs there happen to be more ſurds than one, in which caſe the operation is to be repeated.*

Thus, $\sqrt{x} + 6 = 10$, by tranſpoſition, becomes $\sqrt{x}\ (= 10 - 6) = 4$; which, by ſquaring both ſides, gives $x = 16$.

So, likewiſe, $\sqrt{aa + xx} - c = x$, becomes $\sqrt{aa + xx} = c + x$; which, ſquared, gives $aa + xx = cc + 2cx + xx$, or $aa - cc = 2cx$ (*by rule* 1).—The Reaſons of this, as well as of the two preceding rules, depend on ſelf-evident principles: for, when the equal quantities, on each ſide of an equation, are multiplied or divided by the ſame, or by equal quantities, or raiſed to equal powers, the quantities reſulting muſt neceſſarily be equal.

5°. *Having, by the preceding rules (if there is occaſion) cleared your equation of fractional, and radical quantities, and ſo ordered it, by tranſpoſition, that all the terms, wherein the unknown quantity is found, may ſtand on the ſame ſide thereof, let the whole be divided by the coefficient, or the ſum of the coefficients, of the higheſt power of the ſaid unknown quantity.* And then, if your equation be a ſimple one (that is, if the firſt power or the quantity itſelf, be only concerned) the work is at an end; but if it be a quadratic, or cubic one, &c. ſomething further remains to be done; and recourſe muſt be had to the particular methods for reſolving theſe kinds of equations, hereafter to be conſidered in a proper place.

I ſhall here ſubjoin a few examples for the Learner's exerciſe, wherein all the aforegoing Rules obtain promiſcuouſly.

Ex. 1. Let $5x - 16 = 3x + 12$; then *(by rule* 1) $5x - 3x = 12 + 16$, or $2x = 28$: whence *(by rule* 5) $x = \frac{28}{2} = 14$.

Ex. 2. Let $20 - 3x - 8 = 60 - 7x$: then $-3x + 7x = 60 - 20 + 8$, that is, $4x = 48$; and conſequently $x = \frac{48}{4} = 12$.

Ex. 3. Let $ax - b = cx + d$; then $ax - cx = d + b$, or $\overline{a - c} \times x = d + b$; and therefore $x = \frac{d + b}{a - c}$ *(by rule* 5.)

Ex. 4. Let $6x^2 - 20x = 16x + 2x^2$: then, dividing by $2x$ *(according to rule* 2) we have $3x - 10 = 8 + x$; whence $3x - x = 8 + 10$, that is, $2x = 18$; and therefore $x = \frac{18}{2} = 9$.

Ex. 5. Let $3ax^3 - abx^2 = ax^3 + 2acx^2$: here, dividing the whole by ax^2, we have $3x - b = x + 2c$; therefore $2x = 2c + b$, and $x = \frac{2c + b}{2}$.

Ex. 6. Let $\frac{x}{3} + \frac{x}{4} = 21$: then (*by rule* 3) $4x + 3x = 252$; and therefore $x = \frac{252}{7} = 36$.

Ex. 7. Let $\frac{x + 1}{2} + \frac{x + 2}{3} = 16 - \frac{x + 3}{4}$: then, $12x + 12 + 8x + 16 = 384 - 6x - 18$; whence $26x = 338$, and $x = \frac{338}{26} = 13$.

Ex. 8. Let $a - \frac{bb}{x} = c$: then $ax - bb = cx$; whence $ax - cx = bb$, and $x = \frac{bb}{a - c}$.

Ex. 9. Let $\frac{x}{a} + \frac{x}{b} + \frac{x}{c} = d$: then, $bcx + acx + abx = abcd$, or $\overline{bc + ac + ab} \times x = abcd$; and conſequently $x = \frac{abcd}{bc + ac + ab}$ (*by rule* 5.)

Ex. 10. Let $ax + b^2 = \frac{ax^2 + ac^2}{a + x}$: then, $\overline{ax + b^2} \times \overline{a + x} = ax^2 + ac^2$, that is, $a^2x + ab^2 + ax^2 + b^2x = ax^2 + ac^2$; whence $a^2x + ax^2 + b^2x - ax^2 = ac^2 - ab^2$, or $a^2x + b^2x = ac^2 - ab^2$; and therefore $x = \frac{ac^2 - ab^2}{aa + bb}$.

Ex. 11. Let $\frac{a}{a + x} + \frac{b}{x} = 1$: then, $ax + ab + bx = ax + xx$; whence $-xx + bx = -ab$; which, by changing all the ſigns (in order that the higheſt power of x may be poſitive) gives $xx - bx = ab$. But the ſame concluſion may be otherwiſe brought out, by firſt changing the ſides of the equation $ax + ab + bx = ax + xx$;

$+ ax$; which thereby becoming $ax + xx = ax + ab + bx$, we thence get $xx - bx = ab$, *the ſame as before.*

Ex. 12. Let $\frac{\sqrt{5x}}{3} + 12 = 17$: then $\frac{\sqrt{5x}}{3} = 5$, and $\sqrt{5x} = 15$; whence, (*by rule* 4) $5x = 225$, and therefore $x = \frac{225}{5} = 45$.

Ex. 13. Let $\sqrt{12 + x} = 2 + \sqrt{x}$: then (*by rule* 4) $12 + x = 4 + 4\sqrt{x} + x$; whence, by tranſpoſition, $8 = 4\sqrt{x}$; and, by diviſion, $2 = \sqrt{x}$; conſequently $4 = x$.

Ex. 14. Let $x + \sqrt{a^2 + x^2} = \frac{2a^2}{\sqrt{a^2 + x^2}}$. Here (*by rule* 3) $x \times \sqrt{a^2 + x^2} + a^2 + x^2 = 2a^2$; whence $x \times \sqrt{a^2 + x^2} = a^2 - x^2$, and $x^2 \times \overline{a^2 + x^2} = a^4 - 2a^2x^2 + x^4$ (*by rule* 4), that is, $a^2x^2 + x^4 = a^4 - 2a^2x^2 + x^4$; therefore $3a^2x^2 = a^4$, and $x^2 = \frac{a^4}{3a^2} = \frac{a^2}{3}$.

Ex. 15. Let $\sqrt{x} + \sqrt{a + x} = \frac{2a}{\sqrt{a + x}}$. Then $\sqrt{ax + xx} + a + x = 2a$, or $\sqrt{ax + xx} = a - x$; whence $ax + xx = a^2 - 2ax + x^2$, and $x = \frac{a^2}{3a} = \frac{a}{3}$.

Ex. 16. Let $\sqrt[3]{x^3 - a^3} = x - c$: then, by cubing both ſides, $x^3 - a^3 = x^3 - 3cx^2 + 3c^2x - c^3$; whence $3cx^2 - 3c^2x = a^3 - c^3$, and $x^2 - cx = \frac{a^3}{3c} - \frac{c^2}{3}$ by dividing the whole by $3c$.

Ex. 17. Let $\sqrt{aa + xx} = \sqrt[4]{b^4 + x^4}$: then, by raiſing both ſides to the fourth power, we have $\overline{aa + xx}|^2 = b^4 + x^4$, that is, $a^4 + 2a^2x^2 + x^4 = b^4 + x^4$; and conſequently $x^2 = \frac{b^4 - a^4}{2aa} = \frac{b^4}{2aa} - \frac{1}{2}a^2$.

Ex. 18. Let $x = \sqrt{a^2 + x\sqrt{bb + xx}} - a$. Here $x + a = \sqrt{a^2 + x\sqrt{bb + xx}}$; which ſquared, gives $x^2 + 2ax + a^2 = a^2 + x\sqrt{bb + xx}$, or $x^2 + 2ax = x\sqrt{bb + xx}$; divide by x, ſo ſhall $x + 2a = \sqrt{bb + xx}$; this ſquared again gives $x^2 + 4ax + 4a^2 = bb + xx$; whence $4ax = bb - 4a^2$, and therefore $x = \frac{bb}{4a} - a$.

Of the Extermination of unknown quantities, or the reduction of two or more equations to a ſingle one.

It has been ſhewn above, how to manage a ſingle equation; but it often happens, that, in the ſolution of the ſame problem, two, or three, or more equations are concerned, and as many unknown quantities, mixed promiſcuouſly in each of them; which equations, before any one of thoſe quantities can be known, muſt be reduced into one, or ſo ordered and connected, that, from thence, a new equation may at length ariſe, affected with only one unknown quantity. This, in moſt caſes, may be performed various ways, but the following are the moſt general.

1°. *Obſerve which, of all your unknown quantities, is the leaſt involved, and let the value of that quantity be found in each equation (by the methods already explained) looking upon all the reſt as known; let the values thus found be put equal to each other (for they are equal, becauſe they all expreſs the ſame thing); whence new equations will ariſe, out of which that quantity will be totally excluded; with which new equations the operation may be repeated, and the unknown quantities exterminated, one by one, till, at laſt, you come to an equation containing only one unknown quantity.*

2°. *Or, let the value of the unknown quantity, which you would firſt exterminate, be found in that equation wherein it is the leaſt involved, conſidering all the other quantities as known; and let this value, and its powers, be ſubſtituted for that quantity and its reſpective powers in the* *other equations; and with the new equations thus ariſing repeat the operation, till you have only one unknown quantity, and one equation.*

3°. Or, laſtly, let the given equations be multiplied or divided by ſuch numbers or quantities, whether known or unknown, that the term which involves the higheſt power of the unknown quantity to be exterminated, may be the ſame in each equation; and then, by adding, or ſubtracting the equations, as occaſion ſhall require, that term will vaniſh, and a new equation emerge, wherein the number of dimenſions (if not the number of unknown quantities) will be diminiſhed.

But the uſe of the different methods here laid down will be more clearly underſtood by help of a few examples.

EXAMPLE I.

Let there be given the equations $x + y = 12$, *and* $5x + 3y = 50$; *to find* x *and* y.

According to the firſt Method, by tranſpoſing y and $3y$, we get $x = 12 - y$, and $5x = 50 - 3y$; from the laſt of which equations, $x = \frac{50 - 3y}{5}$: Now, by equating theſe two values of x, we have $12 - y = \frac{50 - 3y}{5}$; and therefore $60 - 5y = 50 - 3y$: from which y is given $= \frac{10}{2} = 5$; and $x\ (= 12 - y = 12 - 5) = 7$.

According to the ſecond Method; x being, by the firſt equation, $= 12 - y$, this value muſt therefore be ſubſtituted in the ſecond, that is, $60 - 5y$ muſt be wrote in the room of its equal $5x$; whence will be had $60 - 5y + 3y = 50$; and from thence $y = \frac{10}{2} = 5$, *as before*.

But according to the third Method, having multiplied the firſt equation by 5, *it* will ſtand thus, $5x + 5y = 60$; from whence ſubtracting the 2d equation, $5x + 3y = 50$, there remains — — — — $2y = 10$; whence $y = 5$, *ſtill the ſame as before*.

The firſt of theſe three ways is much uſed by ſome Authors, but the laſt of them is, for the general part, the moſt eaſy and expeditious in practice, and is, for that reaſon, chiefly regarded in the ſubſequent examples.

EXAMPLE II.

Let $\begin{cases} 5x + 8y = 124 \\ 3x - 2y = 20. \end{cases}$

Here the ſecond equation being multiplied by 4 (in order that the coefficients of y in both equations may be the ſame) we have $12x - 8y = 80$.

Let this equation and the firſt be, now, added together; whence y will be exterminated, there coming out $17x = 204$; from which $x = \frac{204}{17} = 12$: therefore, by the firſt equation, $y \left(= \frac{124 - 5x}{8} = \frac{124 - 60}{8} = \frac{64}{8}\right) = 8$.

EXAMPLE III.

Given $\begin{cases} 5x - 3y = 90 \\ 2x + 5y = 160. \end{cases}$

Here, multiplying the firſt equation by 2, and the ſecond by 5, in order that the coefficient of x may be the ſame in both, there ariſes

$$10x - 6y = 180$$
$$10x + 25y = 800.$$

By ſubtracting the former of which, from the latter, we have $31y = 620$: hence $y = \frac{620}{31} = 20$; and ſo, by the firſt equation, $x \left(= \frac{90 + 3y}{5} = \frac{90 + 60}{5}\right) = 30$.

But the value of x may be otherwiſe found, independent of the value of y; for, by multiplying the firſt equation by 5, and the ſecond by 3, and then adding them together, y will be exterminated, and you will get $25x + 6x = 450 + 480$; whence $x = \frac{930}{31} = 30$, *the ſame as before.*

EXAMPLE IV.

Given $\begin{cases} \frac{x}{2} + \frac{y}{3} = 16 \\ \frac{x}{5} - \frac{y}{9} = 2. \end{cases}$

Here our equations, cleared of fractions, will be

$$3x + 2y = 96$$
$$9x - 5y = 90.$$

And, if from the triple of the former the latter be ſubtracted, we ſhall have $6y + 5y = 288 - 90$, that is, $11y = 198$; whence $y = 18$; and $x \left(= \frac{96 - 2y}{3}\right) = 20$.

EXAMPLE V.

Given $\begin{cases} \frac{x}{2} - 12 = \frac{y}{4} + 8 \\ \frac{x+y}{5} + \frac{x}{3} - 8 = \frac{2y - x}{4} + 27. \end{cases}$

Here $4x - 96 = 2y + 64$, and $12x + 12y + 20x - 480 = 30y - 15x + 1620$; which, contracted, become

$4x - 2y = 160$, and

$47x - 18y = 2100$: from the laſt of which ſubtract 9 times the former; ſo ſhall $11x = 2100 - 1440 = 660$; therefore $x = 60$, and $y \left(= \frac{4x - 160}{2} = 2x - 80\right) = 40$.

EXAMPLE VI.

Let $\left\{ \begin{array}{l} x + y = 13 \\ x + z = 14 \\ y + z = 15 \end{array} \right\}$; to find x, y, and z.

By ſubtracting the firſt equation from the ſecond (in order to extermimate x) we have $z - y = 1$; to which the third equation being added, y will likewiſe be exterminated, there coming out $2z = 16$, or $z = 8$; whence $y (= z - 1) = 7$; and $x (= 13 - y) = 6$.

EXAMPLE VII.

Let $\left\{\begin{aligned} &\frac{x}{2}+\frac{y}{3}+\frac{z}{4}=62 \\ &\frac{x}{3}+\frac{y}{4}+\frac{z}{5}=47 \\ &\frac{x}{4}+\frac{y}{5}+\frac{z}{6}=38. \end{aligned}\right.$

Here the given equations, cleared of fractions, become

$$12x + 8y + 6z = 1488$$
$$20x + 15y + 12z = 2820$$
$$30x + 24y + 20z = 4560,$$

Now (to exterminate z) let the ſecond of theſe equations be ſubtracted from the double of the firſt; and alſo the triple of the third from the quintuple of the ſecond; whence is had

$$4x + y = 156$$
$$10x + 3y = 420:$$

from which $12x - 10x = 468 - 420$, and $x = \frac{48}{2} = 24$.

Therefore $y (= 156 - 4x) = 60$; and $z \left(= \frac{1488 - 8y - 12x}{6}\right)$ $= 120$.

EXAMPLE VIII.

Let $\left\{\begin{aligned} &x + 100 = y + z \\ &y + 100 = 2x + 2z \\ &z + 100 = 3x + 3y. \end{aligned}\right.$

To the double of the firſt, let the ſecond equation be added; ſo ſhall the x's, on the contrary ſides, deſtroy each other, and you will have $300 + y = 2y + 4z$, or $300 = y + 4z$. Moreover, to the triple of the firſt, let the third equation be added, whence will be had $z + 400 = 6y + 3z$, or $400 = 6y + 2z$.

Now, if from the double of this laſt equation, the former, $300 = y + 4z$, be ſubtracted, there will come out $500 = 11y$; and, conſequently, $y = \frac{500}{11} = 45\frac{5}{11}$;

therefore $z\ (= \frac{300-y}{4} = 75 - \frac{y}{4} = 75 - 11\tfrac{4}{11}) = 63\tfrac{7}{11}$; and $x\ (= y + z - 100 = 109\tfrac{1}{11} - 100) = 9\tfrac{1}{11}$.

EXAMPLE IX.

Let $x - y = 2$, and $xy + 5x - 6y = 120$; *to exterminate x.*

By the former equation $x = y + 2$; which value being ſubſtituted in the latter *according to the ſecond general method)* it becomes $\overline{y+2} \times y + 5 \times \overline{y+2} - 6y = 120$, that is, $y^2 + 2y + 5y + 10 - 6y = 120$, or $y^2 + y = 110$.

EXAMPLE X.

Let there be given $x + y = a$, and $x^2 + y^2 = b$; *to exterminate x.*

Here, by the firſt equation, $x = a - y$; and therefore $x^2 = \overline{a-y}|^2$; which value being wrote in the other equation, we have $\overline{a-y}|^2 + y^2 = b$, that is, $a^2 - 2ay + y^2 + y^2 = b$; and therefore $y^2 - ay = \frac{b-aa}{2}$.

EXAMPLE XI.

Given $\left\{ \begin{array}{l} axy + bx + cy = d \\ fxy + gx + hy = k \end{array} \right\}$ *to exterminate y.*

Multiply the firſt equation by f, and the ſecond by a, and ſubtract the latter product from the former; whence you will have $bfx - agx + cfy - ahy = df - ak$; which, by tranſpoſition and diviſion, gives $y = \frac{df-ak+agx-bfx}{cf-ah}$.

Let this value of y be now ſubſtituted in the firſt equation, and there will ariſe

$\frac{adfx - a^2kx + a^2gx^2 - abfx^2 + cdf - cak + cagx - cbfx}{cf - ah} + bx = d$: which, multiplied by $cf - ah$, and contracted, gives $\overline{ag-bf} \times x^2 + \overline{df-ak+cg-bh} \times x = ck - hd$.

EXAMPLE

EXAMPLE XII.

Suppoſing $ax^2+bx+c=0$, and $fx^2+gx+h=0$; *to exterminate* x.

Proceeding here as in the laſt example, we have $fbx+fc-agx-ah=0$; and, from thence, $x=\frac{ah-fc}{fb-ag}$.

Whence, by ſubſtitution, $a\times\frac{\overline{ah-fc}|^2}{\overline{fb-ag}|^2}+\frac{b\times\overline{ah-fc}}{fb-ag}+c=0$. This, by uniting the two laſt terms, and dividing the whole by a, gives $\frac{\overline{ah-fc}|^2}{\overline{fb-ag}|^2}+\frac{bh-cg}{fb-ag}=0$; conſequently $\overline{ah-fc}|^2+\overline{fb-ag}\times\overline{bh-cg}=0$.

After the ſame manner x may be expunged out of the equations $ax^3+bx^2+cx+d=0$, and $fx^2+gx+h=0$, &c. But, to ſhew the uſe of the above example, ſuppoſe there to be given the equations $x^2+yx-y^2=0$, and $x^2+3xy-10=0$: then, by comparing the terms of theſe equations with thoſe of the general ones, $ax^2+bx+c=0$, and $fx^2+gx+h=0$; we have $a=1$, $b=y$, $c=-y^2$, $f=1$, $g=3y$, and $h=-10$; which values being ſubſtituted in the equation $\overline{ah-fc}|^2+\overline{fb-ag}\times\overline{bh-cg}=0$, it thence becomes $\overline{-10+yy}|^2+\overline{y-3y}\times\overline{-10y+3y^3}=0$, that is, $100-20y^2+y^4+20y^2-6y^4=0$; or, $100=5y^4$; whence y may be found, and from thence the value of x alſo.

SECTION X.

Of Proportion.

QUANTITIES, of the ſame kind, may be compared together, either with regard to their differences, or according to the part or parts, that one is of the other, called their ratio. The compariſon of quantities according to their differences, is called *arithmetical*; but according to their ratios, *geometrical.*

When, of four quantities, 2, 6, 12, 16, the difference of the firſt and ſecond is equal to the difference of the third and fourth, thoſe quantities are ſaid to be *in arithmetical proportion.* But, when the ratio of the firſt and ſecond is the ſame with that of the third and fourth (as in 2, 6, 10, 30) then the quantities are ſaid to be in *geometrical proportion.* Moreover, when the difference, or the ratio, of every two adjacent terms (as well of the ſecond and third, as of the firſt and ſecond, &c.) is the ſame, then the proportion is ſaid to be *continued:* thus, 2, 4, 6, 8, &c. is a continued arithmetical proportion; and 2, 4, 8, 16, &c. a continued geometrical one. Theſe kinds of proportions are alſo called Progreſſions, being carried on according to the ſame law throughout.

Arithmetical Proportion.

THEOREM I.

Of any four quantities, a, b, c, d, in arithmetical progreſſion *, *the ſum of the two means is equal to the ſum of the two extremes.*

For ſince, by ſuppoſition, $b - a$ is $= d - c$, therefore is $b + c = d + a$, by tranſpoſition.

THEOREM II.

In any continued arithmetical progreſſion (5, 7, 9, 11, 13, 15) *the ſum of the two extremes and that of every other two terms equally diſtant from them, are equal.*

For ſince, by the nature of Progreſſionals, the ſecond term exceeds the firſt by juſt as much as its correſponding term, the laſt but one, wants of the laſt, it is manifeſt that when theſe correſponding terms are added together, the exceſs of the one will make good the defect of the other, and ſo their ſum be exactly the ſame with that of the two extremes: and in the ſame manner it will appear, that the ſum of any two other correſponding terms muſt be equal to that of the two extremes.

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* Although, in the compariſon of quantities according to their differences, the term *proportion* is uſed; yet the word *progreſſion* is frequently ſubſtituted in its room, and is, indeed, more proper; the former term being, in the common acceptation of it, ſynonymous with *ratio*, which is only uſed in the other kind of compariſon.

When the number of terms is odd, as in the progreſſion, 4, 7, 10, 13, 16, then the ſum of the two extremes being double to the middle term, or mean, the ſum of any other two terms, equally remote from the extremes, muſt likewiſe be double to the mean.

THEOREM III.

In any continued arithmetical progreſſion, $a, a+d, a+2d, a+3d, a+4d,$ *&c. the laſt, or greateſt term, is equal to the firſt (or leaſt), more the common difference of the terms drawn into the number of all the terms after the firſt, or into the whole number of the terms, leſs one.*

For, ſince every term, after the firſt, exceeds that preceding it, by the common difference, it is plain that the laſt muſt exceed the firſt by as many times the common difference as there are terms after the firſt; and therefore muſt be equal to the firſt, and the common difference repeated that number of times.

THEOREM IV.

The ſum of any rank or ſeries of quantities, in continued arithmetical progreſſion (5, 7, 9, 11, 13, 15) *is equal to the ſum of the two extremes multiplied into half the number of terms.*

For, becauſe (by the ſecond Theorem) the ſum of the two extremes, and that of every two other terms equally remote from them, are equal, the whole ſeries, conſiſting of half as many ſuch equal ſums as there are terms, will therefore be equal to the ſum of the two extremes repeated half as many times as there are terms. The ſame thing alſo holds, when the number of terms is odd, as in the ſeries 8, 12, 16, 20, 24; for then, the mean, or middle term, being equal to half the ſum of any two tities according to their differences, is called *arithmetical*; but according to their ratios, *geometrical*.

When, of four quantities, 2, 6, 12, 16, the difference of the firſt and ſecond is equal to the difference of the third and fourth, thoſe quantities are ſaid to be *in arithmetical proportion*. But, when the ratio of the firſt and ſecond is the ſame with that of the third and fourth (as in 2, 6, 10, 30) then the quantities are ſaid to be in *geometrical proportion*. Moreover, when the difference, or the ratio, of every two adjacent terms (as well of the ſecond and third, as of the firſt and ſecond, *&c.*) is the ſame, then the proportion is ſaid to be *continued:* thus, 2, 4, 6, 8, *&c.* is a continued arithmetical proportion; and 2, 4, 8, 16, *&c.* a continued geometrical one. Theſe kinds of proportions are alſo called Progreſſions, being carried on according to the ſame law throughout.

Arithmetical Proportion.

THEOREM I.

*Of any four quantities, a, b, c, d, in arithmetical progreſſion**, *the ſum of the two means is equal to the ſum of the two extremes.*

For ſince, by ſuppoſition, $b - a$ is $= d - c$, therefore is $b + c = d + a$, by tranſpoſition.

THEOREM II.

In any continued arithmetical progreſſion (5, 7, 9, 11, 13, 15) *the ſum of the two extremes and that of every other two terms equally diſtant from them, are equal.*

For ſince, by the nature of Progreſſionals, the ſecond term exceeds the firſt by juſt as much as its correſponding term, the laſt but one, wants of the laſt, it is manifeſt that when theſe correſponding terms are added together, the exceſs of the one will make good the defect of the other, and ſo their ſum be exactly the ſame with that of the two extremes: and in the ſame manner it will appear, that the ſum of any two other correſponding terms muſt be equal to that of the two extremes.

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* Although, in the compariſon of quantities according to their differences, the term *proportion* is uſed; yet the word *progreſſion* is frequently ſubſtituted in its room, and is, indeed, more proper; the former term being, in the common acceptation of it, ſynonymous with *ratio*, which is only uſed in the other kind of compariſon.

When the number of terms is odd, as in the progreſſion, 4, 7, 10, 13, 16, then the ſum of the two extremes being double to the middle term, or mean, the ſum of any other two terms, equally remote from the extremes, muſt likewiſe be double to the mean.

THEOREM III.

In any continued arithmetical progreſſion, a, $a+d$, $a+2d$, $a+3d$, $a+4d$, *&c. the laſt, or greateſt term, is equal to the firſt (or leaſt), more the common difference of the terms drawn into the number of all the terms after the firſt, or into the whole number of the terms, leſs one.*

For, ſince every term, after the firſt, exceeds that preceding it, by the common difference, it is plain that the laſt muſt exceed the firſt by as many times the common difference as there are terms after the firſt; and therefore muſt be equal to the firſt, and the common difference repeated that number of times.

THEOREM IV.

The ſum of any rank or ſeries of quantities, in continued arithmetical progreſſion (5, 7, 9, 11, 13, 15) *is equal to the ſum of the two extremes multiplied into half the number of terms.*

For, becauſe (by the ſecond Theorem) the ſum of the two extremes, and that of every two other terms equally remote from them, are equal, the whole ſeries, conſiſting of half as many ſuch equal ſums as there are terms, will therefore be equal to the ſum of the two extremes repeated half as many times as there are terms. The ſame thing alſo holds, when the number of terms is odd, as in the ſeries 8, 12, 16, 20, 24; for then, the mean, or middle term, being equal to half the ſum of any two terms equally diſtant from it, on contrary ſides, it is obvious that the value of the whole ſeries is the ſame, as if every term thereof was equal to the mean, and therefore is equal to the mean (or half the ſum of the two extremes) multiplied by the whole number of terms; or, to the whole ſum of the extremes multiplied by half the number of terms.

Geometrical Proportion.

THEOREM I.

If four quantities, a, b, c, d, (2, 6, 5, 15) are in geometrical proportion, the product of the two means, bc, will be equal to that of the two extremes, ad,

For, ſince the ratio of a to b (or the part which a is of b) is expreſſed by $\frac{a}{b}$, and the ratio of c to d, in like manner, by $\frac{c}{d}$; and ſince, by ſuppoſition, theſe two ratios are equal, let them both be multiplied by bd, and the products $\frac{a}{b} \times bd$ and $\frac{c}{d} \times bd$ will likewiſe be equal; that is $\frac{abd}{b} = \frac{cbd}{d}$, or $ad = cb$ (by *caſe* 2. *ſect.* 4.)

THEOREM II.

If four quantities, a, b, c, d, are ſuch, that the product of two of them, ad, is equal to the product of the other two, bc, then are thoſe quantities proportional.

For ſince, by ſuppoſition, the products ad and bc are equal, let both be divided by bd, and the quotients $\frac{ad}{bd}$ $\left(\frac{a}{b}\right)$ and $\frac{bc}{bd}$ $\left(\frac{c}{d}\right)$ will alſo be equal; and therefore $a : b :: c : d$.

THEOREM III.

If four quantities, a, b, c, d (2, 6, 5, 15) are proportional, the rectangle of the means divided by either extreme, will give the other extreme.

For, by the ſecond Theorem, $ad = bc$ $(2 \times 15 = 6 \times 5)$, whence dividing both ſides of the equation by a (2), we have $d = \frac{bc}{a}$ $(15 = \frac{6 \times 5}{2})$. Hence, if the two means and one extreme be given, the other extreme may be found.

THEOREM IV.

The products of the correſponding terms of two geometrical proportions, are alſo proportional.

That is, if $a : b :: c : d$, and $e : f :: g : h$, then will $ae : bf :: cg : dh$.

For $\frac{a}{b} = \frac{c}{d}$, and $\frac{e}{f} = \frac{g}{h}$, by ſuppoſition; whence $\frac{a}{b} \times \frac{e}{f} = \frac{c}{d} \times \frac{g}{h}$, by equal multiplication; and conſequently $\frac{ae}{bf} = \frac{cg}{dh}$ (by p. 18); that is, $ae : bf :: cg : dh$.

Hence it follows, that, if four quantities are proportional, their ſquares, cubes, &c. will likewiſe be proportional.

THEOREM V.

If four quantities, a, b, c, d, (2, 6, 5, 15) *are proportional,*

Then,
1. inverſely, $b : a :: d : c$ $(6 : 2 :: 15 : 5)$
2. alternately, $a : c :: b : d$ $(2 : 5 :: 6 : 15)$
3. compoundedly, $a : a+b :: c : c+d$ $(2 : 8 :: 5 : 20)$
4. dividedly, $a : b-a :: c : d-c$ $(2 : 4 :: 5 : 10)$
5. mixtly, $b+a : b-a :: d+c : d-c$ $(8 : 4 :: 20 : 10)$
6. by multiplication, $ra : rb :: c : d$ $(2r : 6r :: 5 : 15)$
7. by diviſion, $\frac{a}{r} : \frac{b}{r} :: c : d$ $(\frac{2}{r} : \frac{6}{r} :: 5 : 15)$

Becauſe the product of the means, in each caſe, is equal to that of the extremes, and therefore the quantities are proportional, by Theorem 2.

THEOREM VI.

If three numbers, a, b, c, (2, 4, 8) be in continued proportion, the ſquare of the firſt will be to that of the ſecond, as the firſt number to the third; that is, $a^2 : b^2 :: a : c$.

For, ſince $a : b :: b : c$, thence will $ac = bb$, by Theorem 1; and therefore $aac = abb$, by equal multiplication; conſequently $a^2 : b^2 :: a : c$, by Theorem 2.

In like manner it may be proved, that, of four quantities continually proportional, the cube of the firſt is to that of the ſecond, as the firſt quantity to the fourth.

THEOREM VII.

In any continued geometrical proportion (1, 3, 9, 27, 81, *&c.) the product of the two extremes, and that of every other two terms, equally diſtant from them, are equal.*

For the ratio of the firſt term to the ſecond, being the ſame as *that* of the laſt but one to the laſt, theſe four terms are in proportion; and therefore, by Theorem 1, the rectangle of the extremes is equal to *that* of their two adjacent terms: and, after the very ſame manner, it will appear, that the rectangle of the third and laſt but two, is equal to *that* of their two adjacent terms, the ſecond and laſt but one; and ſo of the reſt. Whence the truth of the propoſition is manifeſt.

THEOREM VIII.

The ſum of any number of quantities, in continued geometrical proportion, is equal to the difference of the rectangle of the ſecond and laſt terms and the ſquare of the firſt, divided by the difference of the firſt and ſecond terms.

For, let the firſt term of the proportion be denoted by a, the common ratio by r, the number of terms by n, and the ſum of the whole progreſſion by x: then it is manifeſt that the ſecond term will be expreſſed by $a \times r$, or ar; the third by $ar \times r$, or ar^2; the fourth by $ar^2 \times r$, or ar^3, and the nth, or laſt term, by ar^{n-1}; and therefore the proportion will ſtand thus, $a + ar + ar^2 + ar^3 \ldots\ldots + ar^{n-2} + ar^{n-1} = x$; which equation, multiplied by r, gives $ar + ar^2 + ar^3 + ar^4 \ldots\ldots + ar^{n-1} + ar^n = rx$; from which the firſt equation being ſubtracted there will remain $-a + ar^n = rx - x$; whence $x = (\frac{ar^n - a}{r - 1} = \frac{r \times ar^{n-1} - a}{r - 1}) = \frac{ar \times ar^{n-1} - aa}{ar - a}$; as was to be demonſtrated.

SECTION XI.

The application of Algebra to the reſolution of numerical Problems.

WHEN a Problem is propoſed to be ſolved algebraically, its true deſign and ſignification ought, in the firſt place, to be perfectly underſtood, ſo that (if needful) it may be abſtracted from all ambiguous and unneceſſary phraſes, and the conditions thereof exhibited in the cleareſt light poſſible. This being done, and the ſeveral quantities therein concerned being denoted by proper ſymbols, let the true ſenſe and meaning of the queſtion be tranſlated from the verbal, to a ſymbolical form of expreſſion; and the conditions, thus expreſſed in algebraic terms, will, if *it* be properly limited, give as many equations as are neceſſary to its ſolution. But, if ſuch equations cannot be derived without ſome previous operations (which frequently happens to be the caſe), then let the Learner obſerve this rule, *viz* let him conſider what method or proceſs he would uſe to prove, or ſatisfy himſelf in, the truth of the ſolution, were the numbers that anſwer the conditions of the queſtion to be given, or affirmed to be ſo and ſo; and then, by following the very ſame ſteps, only uſing unknown ſymbols inſtead of known numbers, the queſtion will be brought to an equation.

Thus, if the queſtion were to find a number, which being multiplied by 5, and 8 ſubtracted from the product,

duct, the ſquare of the remainder ſhall be 144; then, having put $a = 5$, $b = 8$, and $c = 144$, ſuppoſe the number ſought

to be	4	(or) x
then 5, or a times that number will be	20	ax
from which 8, or b being ſubtracted, there remains	12	$ax - b$
which, ſquared, is	144	$a^2x^2 - 2axb + b^2$.

Therefore $a^2x^2 - 2axb + b^2$ is $= c$ (or 144) according to the conditions of the queſtion. In the ſame manner may a queſtion be brought to an equation when two or more quantities are required.

After the conditions of a problem are noted down in algebraic terms, the next thing to be done, is to conſider whether it be properly limited, or admits of an indefinite number of anſwers; in order to diſcover which, obſerve the following rules.

RULE I.

When the number of quantities ſought, exceeds the number of equations given, the queſtion (for the general part) is capable of innumerable anſwers.

Thus, if it be required to find two numbers (x and y) with this one ſingle condition, that their ſum ſhall be 100, we ſhall have only one equation, *viz.* $x + y = 100$, but two unknown quantities, x and y, to be determined; therefore it may be concluded, that the queſtion will admit of innumerable anſwers.

RULE II.

But if the number of equations, given from the conditions of the queſtion, is juſt the ſame as the number of quantities ſought, then is the queſtion truly limited.

As, if the queſtion were to find two numbers, whoſe ſum is 100, and whoſe difference is 20; then, x being put for the greater number, and y for the leſs, we ſhall have $x + y = 100$, and $x - y = 20$: therefore, there being here two equations and two unknown quantities, the queſtion is truly limited; 60 and 40 being the only two numbers that can anſwer the conditions thereof.

RULE III.

When the number of equations exceeds the number of quantities ſought, either, the conditions of the problem are inconſiſtent one with another, or what is propoſed, in general terms, can only be poſſible in certain particular caſes.

But it is to be obſerved, that the equations underſtood here, as well as in the preceding rules, are ſuppoſed to be no ways dependent upon, or conſequences of one another. If this be not the caſe, the queſtion may be either unlimited, or abſurd, or perhaps both, at the ſame time that it ſeems truly limited; as will appear by the following example.

Wherein it is required to find three numbers, under theſe conditions; that the ſum of once the firſt, twice the ſecond, and three times the third, may be equal to a given number b; that the ſum of four times the firſt, five times the ſecond, and ſix times the third, may be equal to a given number c; and that the ſum of ſeven times the firſt, eight times the ſecond, and nine times the third, may be equal to a third given number d. Now, the three numbers ſought being, reſpectively, denoted by x, y, and z, the queſtion, in algebraic terms, will ſtand thus,

$$\begin{aligned} x + 2y + 3z &= b \\ 4x + 5y + 6z &= c \\ 7x + 8y + 9z &= d. \end{aligned}$$

Here, there being three equations and juſt the ſame number of unknown quantities, one might conclude the queſtion to be truly limited: but, by reflecting a little upon the nature and form of theſe equations, the contrary will ſoon appear; becauſe the laſt of them includes no new condition but what is compriſed in, and may be derived from the other two; for, if from the double of the ſecond, the firſt equation be taken away, the value of $7x + 8y + 9z$ will from thence be given $= 2c - b$. Hence it is manifeſt, that giving the value of $7x + 8y + 9z$, in the third equation, contributes nothing towards limiting the problem; and that the problem itſelf is not only unlimited, but alſo impoſſible, except when d is given equal to $2c - b$.

Having laid down the neceſſary rules, for bringing problems to equations, and for diſcovering when they are truly limited, it remains that we illuſtrate what is hitherto delivered by proper examples.

Arithmetical Problems.

PROBLEM I.

To find that number, to which 75 being added, the ſum ſhall be the quadruple of the ſaid required number.

Let the number ſought be repreſented by . . . x, then will its quadruple be denoted by . . . . . $4x$; whence, by the conditions of the queſtion, $x + 75 = 4x$; this equation, by tranſpoſing x, becomes . . $75 = 3x$: from whence, dividing by 3, we have $x = \frac{75}{3} = 25$, which is the number that was to be found (for it is plain that $25 + 25 \times 3 = 25 \times 4 = 100$).

PROBLEM II.

What number is that, which being added to 4, and alſo multiplied by 4, the product ſhall be the triple of the ſum?

Let the number ſought be denoted by . . . . x; ſo ſhall the ſum be denoted by . . . . . . $x+4$, and the product by . . . . $4x$: whence, by the conditions of the queſtion, $4x = \overline{x+4} \times 3$; that is, $4x = 3x + 12$; from which by tranſpoſition, $x = 12$.

PROBLEM III.

To find two numbers ſuch, that their ſum ſhall be 30, and their difference 12.

If x be taken to denote the leſſer of the two numbers; then, by adding the difference 12, the greater number will be denoted by $x + 12$; and ſo we ſhall have $2x + 12 = 30$, *by the queſtion.*

From which equation, $2x = 30 - 12 = 18$; and conſequently $x = \frac{18}{2} = 9$: whence the greater number $(x+12)$ is alſo given $= 21$.

PROBLEM IV.

To divide the number 60 *into three ſuch parts, that the firſt may exceed the ſecond by* 8, *and the third by* 16.

Let the firſt part be denoted by x; then the ſecond will be $x - 8$, and the third $x - 16$: the aggregate of all which, or $3x - 24$ is $= 60$, *by the queſtion.*

Hence $3x = 60 + 24 = 84$, and $x = \frac{84}{3} = 28$: ſo that 28, 20, and 12, are the three parts required.

PROBLEM V.

The ſum of 660 *l. was raiſed (for a certain purpoſe) by four perſons,* A, B, C, *and* D: *whereof* B *advanced twice as much as* A; C *as much as* A *and* B; *and* D *as much as* B *and* C: *what did each perſon contribute?*

Let the ſum or number of pounds advanced by A, be called - - - - - - - - - x;
then will the number of B's pounds be denoted by $2x$,
that of C's by - - - - - - - - - - - - $3x$,
and that of D's by - - - - - - - - - - - $5x$;
the ſum of all which is given equal to 660*l.* that is, $11x = 660$: from whence $x = \frac{660}{11} = 60$. Therefore 60, 120, 180, and 300*l.* are the reſpective ſums that were to be determined.

PROBLEM VI.

A certain ſum of money was ſhared among five perſons, A, B, C, D, *and* E; *whereof* B *received* 10*l. leſs than* A; C 16*l. more than* B; D 5*l. leſs than* C; E 15*l. more than* D: *moreover it appeared, that the ſhares of the two laſt together were equal to the ſum of the ſhares of the other three: What was the whole ſum ſhared, and how much did each receive?*

Let x denote the ſhare of A;

$$\text{then } \left\{\begin{array}{l} x - 10 \\ x + 6 \\ x + 1 \\ x + 16 \end{array}\right\} \text{ will be the ſhare of } \left\{\begin{array}{l} \text{B,} \\ \text{C,} \\ \text{D,} \\ \text{E;} \end{array}\right.$$

and therefore $2x + 17 = 3x - 4$, *by the queſtion:* from whence, by tranſpoſition, $21 = x$; ſo that 21, 11, 27, 22, and 37*l.* are the ſeveral required ſhares; amounting, in the whole, to 118*l.*

PROBLEM VII.

To find three numbers, on theſe conditions, that the ſum of the firſt and ſecond ſhall be 15; *of the firſt and third* 16; *and of the ſecond and third* 17.

If the firſt number be denoted by x; then it is plain, by the queſtion, that the ſecond will be repreſented by $15 - x$, and the third by $16 - x$. But the ſum of theſe two laſt is given equal to 17; that is, $31 - 2x = 17$; whence, by tranſpoſition, $14 = 2x$; and conſequently $x = \frac{14}{2} = 7$. Hence $15 - x = 8$, and $16 - x = 9$; which are the other two numbers required.

PROBLEM VIII.

To find that number, which being doubled, and 16 *ſubtracted from the product, the remainder ſhall as much exceed* 100 *as the required number itſelf is leſs than* 100.

The number ſought being denoted by x, the double thereof will be repreſented by $2x$; from which ſubtracting 16, the remainder will be $2x - 16$; and its exceſs above 100, equal to $2x - 16 - 100$: therefore $2x - 16 - 100 = 100 - x$, *by the queſtion*; whence $3x = 216$; and conſequently $x = \frac{216}{3} = 72$.

PROBLEM IX.

To divide the number 75 *into two ſuch parts, that three times the greater may exceed ſeven times the leſſer by* 15.

Let the greater part be $= x$; then will the leſſer part $= 75 - x$, and we ſhall have $3x - 15 = \overline{75 - x} \times 7$; or, which is the ſame, $3x - 15 = 525 - 7x$: from whence $10x = 540$, and conſequently $x = 54$.

PROBLEM X.

Two perſons, A and B, having received equal ſums of money, A out of his paid away 25l. and B of his 60l. and then it appeared that A had juſt twice as much money as B: what money did each receive?

Suppoſe x to denote the ſum received by each perſon; then A, after paying away 25*l.* had $x - 25$; and B, after paying away 60*l.* had $x - 60$: hence $x - 25 = 2x - 120$, *by the queſtion*; and therefore, $120 - 25 = 2x - x$, that is, $95 = x$.

PROBLEM XI.

To find that number, whoſe $\frac{1}{3}$ *part exceeds its* $\frac{1}{4}$ *part by 12.*

Let the number ſought be repreſented by x; then will $\frac{x}{3} - \frac{x}{4} = 12$, by the conditions of the problem; which equation (by multiplying every numerator into all the denominators, except its own) gives $4x - 3x = 144$, that is, $x = 144$.

PROBLEM XII.

What ſum of money is that whoſe $\frac{1}{3}$ *part,* $\frac{1}{4}$ *part, and* $\frac{1}{5}$ *part, added together, ſhall amount to 94 pounds?*

If x be the number of pounds required, then will $\frac{x}{3} + \frac{x}{4} + \frac{x}{5} = 94$: from whence, by reduction, $20x + 15x + 12x = 94 \times 60$, that is, $47x = 94 \times 60$; and therefore $x = 2 \times 60 = 120$.

PROBLEM XIII.

In a mixture of copper, tin, and lead, one half of the whole − 16 lb. *was copper; one third of the whole* − 12 lb. *tin; and one fourth of the whole* + 4 lb. *lead: what quantity of each was there in the compoſition?*

Let x denote the weight of the whole;

$$\text{then will} - \left\{\begin{array}{l}\frac{x}{2}-16\\ \frac{x}{3}-12\\ \frac{x}{4}+4\end{array}\right\} \begin{array}{c}\text{be the weight}\\ \text{of the}\end{array} \left\{\begin{array}{l}\text{copper,}\\ \text{tin,}\\ \text{lead;}\end{array}\right.$$

and, if all theſe be added together, we ſhall have $\frac{x}{2}+\frac{x}{3}+\frac{x}{4}-24=x$, *by the queſtion*. Hence by reduction, $12x+8x+6x-576=24x$; therefore $2x=576$, and $x=\frac{576}{2}=288$. So that there were 128 lb. of copper, 84 lb. of tin, and 76 lb of lead.

PROBLEM XIV.

What ſum of money is that, from which 5l. *being ſubtracted, two thirds of the remainder ſhall be* 40l.?

Let x repreſent the required ſum; then, 5 being ſubtracted, there will remain $x-5$; two thirds of which will be $\overline{x-5}\times\frac{2}{3}$, or $\frac{2x-10}{3}$; and ſo, *by the queſtion*, we have $\frac{2x-10}{3}=40$: whence $2x-10=120$; and $x=\frac{130}{2}=65$.

PROBLEM XV.

What number is that, which being divided by 12, *the quotient, dividend, and diviſor, added all together, ſhall amount to* 64?

Let $x=$ the required number; ſo ſhall $\frac{x}{12}+x+12=64$, by the conditions of the queſtion. Whence $x+12x=52\times12$, or $13x=624$; and conſequently $x=\frac{624}{13}=48$.

PROBLEM XVI.

To find two numbers in the proportion of 2 to 1, ſo that, if 4 be added to each, the two ſums thence ariſing ſhall be in proportion as 3 to 2.

Let x denote the leſſer number; then the greater will be denoted by $2x$; and ſo, *by the queſtion*, we ſhall have $2x + 4 : x + 4 :: 3 : 2$. From whence, as the product of the two extremes, of any four proportional numbers, is equal to the product of the two means, (*ſee Section* 10. *Theorem* 1.) we have the following equation, viz. $\overline{2x + 4} \times 2 = \overline{x + 4} \times 3$, that is, $4x + 8 = 3x + 12$; whence $x = 4$, and $2x = 8$: which are the two numbers that were to be found.

PROBLEM XVII.

A prize of 2000 l. *was divided between two perſons, whoſe ſhares therein were in proportion as* 7 *to* 9: *what was the ſhare of each?*

If $x =$ the ſhare of the firſt, then *that* of the ſecond will be $2000 - x$; and we ſhall have $x : 2000 - x :: 7 : 9$. Hence, by multiplying the extremes and means, $9x = 14000 - 7x$; from which x is found $= \frac{14000}{16} =$ 875*l.* and $2000 - x =$ 1125*l.*

PROBLEM XVIII.

A bill of 120 l. *was paid in guineas and moidores, and the number of pieces of both ſorts was juſt* 100; *to find how many there were of each.*

If $x =$ the number of guineas, then will $100 - x$ be the number of moidores: therefore, the number of ſhillings in the guineas being $21x$, and, in the moidores, $27 \times \overline{100 - x}$, we have $21x + 27 \times \overline{100 - x} = 120 \times 20 =$ the ſhillings in the whole ſum: hence, by multiplication, $21x + 2700 - 27x = 2400$; and $x = \frac{300}{6} = 50$.

PROBLEM XIX.

A labourer engaged to ſerve 40 days, on theſe conditions, that for every day he worked he was to receive 20 pence, but that, for every day he played, or was abſent, he was to forfeit 8 pence; now after the 40 days were expired, it was found that he had to receive, upon the whole, 380 pence: the queſtion is, to find how many days of the forty he worked, and how many he played.

Let the number expreſſing the days he worked, be repreſented by x; then the number of days he played will be expreſſed by $40 - x$: moreover, ſince he was to receive 20 pence for every day he worked, the whole number of pence gained by working, will be $20x$; and for the like reaſon, the number of pence forfeited by playing, or being abſent, will be $8 \times \overline{40 - x}$, or $320 - 8x$; which deducted from $20x$, leaves $28x - 320$, for the ſum total of what he had to receive: whence we have this equation, $28x - 320 = 380$; from which $28x = 380 + 320 = 700$, and conſequently $x = \frac{700}{28} = 25$, equal to the number of days he worked; therefore $40 - 25 = 15$, will be the number of days he played.

PROBLEM XX.

A farmer would mix two ſorts of grain, viz. wheat, worth 4s. *a buſhel, with rye, worth* 2s. 6d. *the buſhel, ſo that the whole mixture may conſiſt of* 100 *buſhels, and be worth* 3s. *and* 2d. *the buſhel: now it is required to find how many buſhels of each ſort muſt be taken to make up ſuch a mixture.*

Let the number of buſhels of wheat be put $= x$, and the number of buſhels of rye will be $100 - x$: but the number of buſhels multiplied by the number of pence *per* buſhel, is equal to the number of pence the whole is worth; therefore $48x$ is the whole value of the wheat, and $30 \times \overline{100 - x}$, or $3000 - 30x$, that of the rye; and conſequently, $48x + 3000 - 30x$, the ſum of theſe two, the whole value of the mixture: which, *by the queſtion*, is equal to 100×38, or 3800 pence: hence we have $48x + 3000 - 30x = 3800$; and therefore, $x = \frac{800}{18} = 44\frac{4}{9}$, the number of buſhels of wheat; whence the number of buſhels of rye will be $100 - 44\frac{4}{9} = 55\frac{5}{9}$.

PROBLEM XXI.

A farmer ſold, to one man, 30 *buſhels of wheat and* 40 *of barley, and for the whole received* 270 *ſhillings; and to another he ſold* 50 *buſhels of wheat and* 30 *of barley, at the ſame prices, and for the whole received* 340 *ſhillings: now it is required to find what each ſort of grain was ſold at per buſhel.*

Let x and y be, reſpectively, the number of ſhillings which a buſhel of each ſort was ſold for; then, from the conditions of the queſtion, we ſhall have theſe two equations, *viz.*

$$30x + 40y = 270,$$
$$50x + 30y = 340;$$

from 4 times the ſecond of which ſubtract 3 times the firſt, ſo ſhall $110x = 550$; and conſequently $x = \frac{550}{110} = 5$: moreover, by ſubtracting 3 times the ſecond, from 5 times the firſt, you will have $110y = 330$, and therefore $y = \frac{330}{110} = 3$.

$$\text{For} \begin{cases} 30 \times 5 + 40 \times 3 = 270, \\ 50 \times 5 + 30 \times 3 = 340. \end{cases}$$

PROBLEM XXII.

A ſon aſking his father how old he was, received the following reply; My age, ſays the father, 7 *years ago, was juſt four times as great as yours at that time; but* 7 *years hence, if you and I live, my age will then be only double to yours: it is required to find from hence, the age of each perſon.*

Let x repreſent the age of the ſon ſeven years before the queſtion; then the age of the father, at that time, was $4x$, by the conditions of the queſtion; and, if each of theſe ages be increaſed by 14, it is plain that $x + 14$ and $4x + 14$ will reſpectively expreſs the two ages 7 years after the time in queſtion; whence, again, by the problem, we have $4x + 14 = 2 \times \overline{x + 14}$; from which $x = 7$, and $4x = 28$; therefore $7 + 7 = 14$, and $28 + 7 = 35$, are the two ages required.

For $\begin{cases} 35 - 7 = \overline{14 - 7} \times 4, \\ 35 + 7 = \overline{14 + 7} \times 2. \end{cases}$

PROBLEM XXIII.

A gentleman hired a ſervant for 12 *months, and agreed to allow him* 20l. *and a livery, if he ſtaid till the year was expired; but at the end of* 8 *months the ſervant went away and received* 12l. *and the livery, as a proportional part of his wages: the queſtion is, what was the livery valued at?*

Let x be the value ſought; then $20 + x$ will be the whole wages for 12 months, and $12 + x$ the part thereof which he received for 8 months.

But, the wages being in the ſame proportion as the times in which they are earned, or become due, we therefore have, as $12 : 8 :: 20 + x : 12 + x$; whence $12 \times \overline{12 + x} = 8 \times \overline{20 + x}$, or $144 + 12x = 160 + 8x$, (by Theor. 1, p. 72) conſequently $12x - 8x = 160 - 144$, and $x = \frac{16}{4} = 4l.$

PROBLEM XXIV.

Four perſons, A, B, C, D, *ſpent twenty ſhillings in company together; whereof* A *propoſed to pay* $\frac{1}{3}$; B $\frac{1}{4}$; C $\frac{1}{5}$; *and* D $\frac{1}{6}$ *part; but, when the money came to be collected, they found it was not ſufficient to anſwer the intended purpoſe: the queſtion then is, to find how much each perſon muſt contribute, to make up the whole reckoning, ſuppoſing their ſeveral ſhares to be to each other in the proportion above ſpecified?*

Let x be the ſhare of A; then it will be, as $\frac{1}{3} : \frac{1}{4}$, or, as $4 : 3 :: x : \frac{3x}{4}$ = the ſhare of B; and, as

$\frac{1}{3} : \frac{1}{5}$, or, as $5 : 3 :: x : \frac{3x}{5} =$ the ſhare of C; alſo, as $\frac{1}{3} : \frac{1}{6}$, or, as $2 : 1 :: x : \frac{x}{2} =$ the ſhare of D.

Therefore, *by the queſtion*, $x + \frac{3x}{4} + \frac{3x}{5} + \frac{x}{2} = 20$; whence, $40x + 30x + 24x + 20x = 800$, that is, $114x = 800$; and conſequently $x = \frac{800}{114} = 7\frac{1}{57}$, the ſhare of A; therefore $(\frac{3x}{4})$ that of B will be $= 5\frac{15}{57}$: that of C $(\frac{3x}{5}) = 4\frac{12}{57}$; and that of D $(\frac{x}{2}) = 3\frac{20}{57}$.

PROBLEM XXV.

A market-woman bought in a certain number of eggs, at 2 a penny, and as many at 3 a penny, and ſold them all out again, at the rate of 5 for two pence, and loſt four pence by ſo doing: what number of eggs did ſhe buy and ſell?

Let x be the number of eggs of each price, or ſort; then $\frac{x}{2}$ will be the number of pence which all the firſt ſort coſt, and $\frac{x}{3}$ the price of all the ſecond ſort; but the whole price of both ſorts together, at the rate of 5 for two pence, at which they were ſold, will be $\frac{4x}{5}$ (for as $5 : 2 :: 2x$ (the whole number of eggs) $: \frac{4x}{5}$) hence, *by the queſtion*, $\frac{x}{2} + \frac{x}{3} - \frac{4x}{5} = 4$; whence $15x + 10x - 24x = 120$, and therefore $x = 120$.

For $\frac{120}{2} + \frac{120}{3} - \frac{240}{5} \times 2 = 60 + 40 - 96 = 4$.

PROBLEM XXVI.

A compoſition of copper and tin, containing 100 cubic inches, being weighed, its weight was found to be 505 ounces:

how many ounces of each metal did it contain, ſuppoſing the weight of a cubic inch of copper to be 5¼ ounces, and that of a cubic inch of tin 4¼?

Let x be the number of ounces of copper; then $505 - x$ will be the number of ounces of tin, and we ſhall have

$5\frac{1}{4} : 1$ (cubic inch) $: : x : \frac{x}{5\frac{1}{4}}$ inches of copper,

$4\frac{1}{4} : 1$ (cubic inch) $: : 505 - x : \frac{505 - x}{4\frac{1}{4}}$ inches of tin.

Therefore $\frac{x}{5\frac{1}{4}} + \frac{505 - x}{4\frac{1}{4}} = 100$, *by the queſtion.* Whence $4\frac{1}{4} \times x + 5\frac{1}{4} \times \overline{505 - x} = 5\frac{1}{4} \times 4\frac{1}{4} \times 100$, that is, $\frac{17 \times x}{4} + \frac{21 \times \overline{505 - x}}{4} \left(= \frac{21 \times 17 \times 100}{4 \times 4}\right) = \frac{21 \times 17 \times 25}{4}$; which, by rejecting the common diviſor, becomes $17x + 21 \times \overline{505 - x} = 21 \times 17 \times 25 = 8925$, or $17x - 21x = 8925 - 10605 = -1680$. From whence $x = \frac{1680}{4} = 420$; and $505 - x = 85$; which are the two numbers required.

The ſame otherwiſe.

Suppoſe x to be the number of ſolid inches of copper; then the number of inches of tin being $100 - x$, we have $5\frac{1}{4} \times x + 4\frac{1}{4} \times \overline{100 - x} = 505$, that is $5\frac{1}{4}x + 425 - 4\frac{1}{4}x = 505$, or $x = 505 - 425 = 80$; which, multiplied by $5\frac{1}{4}$, gives 420, for the ounces of copper.

PROBLEM XXVII.

A ſhepherd, in time of war, fell in with a party of ſoldiers, who plundered him of half his flock, and half a ſheep over; afterwards a ſecond party met him, who took half what he had left, and half a ſheep over; and, ſoon after this, a third party met him, and uſed him in the ſame manner; and then he had only five ſheep left: it is required to find what number of ſheep he had at firſt.

Let x (as uſual) be the number ſought; then, according to the queſtion, the number of ſheep left, after being plundered the firſt time, will be expreſſed by $\frac{x}{2} - \frac{1}{2}$, or $\frac{x-1}{2}$; the half of which is $\frac{x-1}{4}$; from whence ſubtracting $\frac{1}{2}$, the remainder $(\frac{x-1}{4} - \frac{1}{2})$ $\frac{x-3}{4}$ will be the number of ſheep left after being plundered the ſecond time: in like manner, if from $\frac{x-3}{8}$ (the half of $\frac{x-3}{4}$) you, again, take $\frac{1}{2}$, there will remain $\frac{x-3}{8} - \frac{1}{2}$ or $\frac{x-7}{8}$, the number of ſheep remaining at laſt. Hence we have $\frac{x-7}{8} = 5$; therefore $x - 7 = 40$, and $x = 47$.

PROBLEM XXVIII.

The difference of two numbers being given, equal to 4, and the difference of their ſquares, equal to 40; to find the numbers.

Let the leſſer number be x; then, the difference being 4, the greater muſt conſequently be $x + 4$, and its ſquare $xx + 8x + 16$, from which xx, the ſquare of the leſſer being taken away, the difference is $8x + 16$: therefore $8x + 16 = 40$; which, reduced, gives $x = 3$; whence $x + 4 = 7$; therefore the two required numbers are 3 and 7.

All the problems hitherto delivered are reſolved by a *numeral exegeſis*, wherein the unknown quantities, only, are repreſented by letters of the alphabet; which ſeemed neceſſary, in order to ſtrengthen the Beginner's idea, at ſetting out, and lead him on by proper gradations: but it is not only more maſterly and elegant, but alſo more uſeful, to repreſent the known, as well as the unknown quantities, by algebraic ſymbols; ſince from thence a general theorem is derived, whereby all other queſtions of the ſame kind may be reſolved.

As an inſtance hereof, let the laſt problem be again reſumed; then, the given difference of the required numbers being denoted by a, the difference of their ſquares by b, and the leſſer number by x; the greater will be $x + a$, and its ſquare $x^2 + 2xa + a^2$; from which, x^2, the ſquare of the leſſer number, being deducted, there remains $2xa + a^2 = b$: whence, if aa be ſubtracted from both ſides, there will remain $2ax = b - aa$; this, divided by $2a$, gives $x = \frac{b}{2a} - \frac{a}{2}$; and conſequently $x + a = \frac{b}{2a} + \frac{a}{2}$. Hence it appears, that, if the difference of the ſquares be divided by twice the difference of the numbers, and half the difference of the numbers be ſubtracted from the quotient, the remainder will be the leſſer number; but if half the difference of the numbers be added to the quotient, the ſum will give the greater number. Thus, if the difference (a) be 4, and the difference (b) of the ſquares 40 (as in the caſe above); then ($\frac{b}{2a}$) the difference of the ſquares, divided by twice the difference of the numbers, will be 5; from which ſubtracting (2) half the difference of the numbers, there remains 3, for the leſſer number ſought; and, by adding the ſaid half difference, you will have 7 = the greater number. In the ſame manner, if the difference of the two numbers had been given 6, and the difference of their ſquares 60, the numbers themſelves would have come out 2 and 8: and ſo of any other.

PROBLEM XXIX.

Having given the ſum of two numbers, equal to 30, *and the difference of their ſquares, equal to* 120; *to find the numbers.*

Put $a = 30$, and $b = 120$, and let x be the leſſer number ſought, and then the greater will be $a - x$; whoſe ſquare is $aa - 2ax + x^2$; from which the ſquare of the leſſer being ſubtracted, we have $a^2 - 2ax = b$; this reduced, gives x, the leſſer number, $= \frac{a}{2} - \frac{b}{2a} = 13$. Therefore the greater $(a - x)$ will be $= a - \frac{a}{2} + \frac{b}{2a} = \frac{a}{2} + \frac{b}{2a} = 17$. But if the greater number had been firſt made the object of our enquiry, or been put $= x$, the leſſer would have been $a - x$, and its ſquare $a^2 - 2ax + x^2$, which ſubtracted from x^2 leaves $2ax - a^2 = b$; whence $2ax = b + a^2$, and $x = \frac{b}{2a} + \frac{a}{2}$ $= 17$, the ſame as before.

PROBLEM XXX.

If one agent A, *alone, can produce an effect* e, *in the time* a, *and another agent* B, *alone, in the time* b; *in how long time will they both together produce the ſame effect?*

Let the time ſought be denoted by x, and it will be, as $a : x :: e : \frac{ex}{a}$, the part of the effect produced by A: (*Theor.* 3. p. 72) alſo, as $b : x :: e : \frac{ex}{b}$, the part produced by B: therefore $\frac{ex}{a} + \frac{ex}{b} = e$. Divide the whole by e, and you will have $\frac{x}{a} + \frac{x}{b} = 1$; and this, reduced, gives $x = \frac{ab}{a + b}$. After the ſame manner, if there be three agents, A, B, and C, the time wherein they will all together produce the given effect, will come out $= \frac{abc}{ab + ac + bc}$.

Example. Suppoſe A, alone, can perform a piece of work in 10; B, alone, in 12 days; and C, alone, in 1[illegible] days; then all three together will perform the ſame piece of work in $4\frac{4}{59}$ days; for in this caſe, a being $= 10$, $b = 12$, $c = 16$, it is plain that $\frac{abc}{ab + ac + bc}$ $\left(\frac{10 \times 12 \times 16}{10 \times 12 + 10 \times 16 + 12 \times 16}\right) = 4\frac{4}{59}$.

PROBLEM XXXI.

Two travellers, A *and* B, *ſet out together from the ſame place, and travel both the ſame way;* A *goes* 28 *miles the firſt day,* 26 *the ſecond,* 24 *the third, and ſo on, decreaſing two miles every day; but* B *travels uniformly* 20 *miles every day: now it is required to find how many miles each perſon muſt travel before* B *comes up again with* A?

Let $x =$ the number of days in which B overtakes A: then the miles travelled by B, in that time, will be $20x$; and thoſe travelled by A, $28 + 26 + 24 + 22$, &c. continued to x terms; where the laſt term (*by Section* 10. *Theorem* 3) will be equal to $28 - 2 \times \overline{x - 1}$, or $30 - 2x$; and therefore the ſum of the whole progreſſion equal to $\overline{28 + 30 - 2x} \times \frac{1}{2}x$, or $29x - x^2$ (*by Theorem* 4.) Hence we have $20x = 29x - x^2$; whence $20 = 29 - x$, and $x = 9$: therefore $20 \times 9 = 180$ is the diſtance which was to be found.

PROBLEM XXXII.

To find three numbers, ſo that $\frac{1}{2}$ *the firſt,* $\frac{1}{3}$ *of the ſecond, and* $\frac{1}{4}$ *of the third, ſhall be equal to* 62; $\frac{1}{3}$ *of the firſt,* $\frac{1}{4}$ *of the ſecond, and* $\frac{1}{5}$ *of the third, equal to* 47; *and* $\frac{1}{4}$ *of the firſt,* $\frac{1}{5}$ *of the ſecond, and* $\frac{1}{6}$ *of the third, equal to* 38.

Put $a = 62$, $b = 47$, and $c = 38$, and let the numbers ſought be denoted by x, y, and z; then the conditions of the problem, expreſſed in algebraic terms, will ſtand thus,

$$\frac{x}{2} + \frac{y}{3} + \frac{z}{4} = a,$$

$$\frac{x}{3} + \frac{y}{4} + \frac{z}{5} = b,$$

$$\frac{x}{4} + \frac{y}{5} + \frac{z}{6} = c.$$

Which, cleared of fractions, become

$$\begin{aligned} 6x + 4y + 3z &= 12a, \\ 20x + 15y + 12z &= 60b, \\ 15x + 12y + 10z &= 60c. \end{aligned}$$

And, by ſubtracting the ſecond of theſe equations from the quadruple of the firſt, (in order to exterminate z) we have $4x + y = 48a - 60b$; moreover, by taking 3 times the third from 10 times the firſt, we have $15x + 4y = 120a - 180c$; this ſubtracted from 4 times the laſt, leaves $x = 72a - 240b + 180c = 24$; whence $y\ (48a - 60b - 4x) = 60$, and $z\ (\frac{12a - 6x - 4y}{3}) = 120$.

$$\text{For} \begin{cases} \frac{24}{2} + \frac{60}{3} + \frac{120}{4} = 12 + 20 + 30 = 62, \\ \frac{24}{3} + \frac{60}{4} + \frac{120}{5} = 8 + 15 + 24 = 47, \\ \frac{24}{4} + \frac{60}{5} + \frac{120}{6} = 6 + 12 + 20 = 38. \end{cases}$$

PROBLEM XXXIII.

A gentleman left a ſum of money to be divided among four ſervants, ſo that the ſhare of the firſt was ½ the ſum of the ſhares of the other three; the ſhare of the ſecond ⅓ of the ſum of the other three; and the ſhare of the third ¼ of the ſum of the other three; and it was alſo found that the ſhare of the firſt exceeded that of the laſt by 14l: *the queſtion is, what was the whole ſum, and what was the ſhare of each perſon?*

Let the ſhares be repreſented by x, y, z, and u, reſpectively, and let $a = 14$; then, *by the queſtion*, we ſhall have

$$\begin{aligned} x &= \frac{y + z + u}{2}, \\ y &= \frac{x + z + u}{3}, \\ z &= \frac{x + y + u}{4}, \\ u &= x - a. \end{aligned}$$

Which equations, cleared of fractions, become

$$2x = y + z + u,$$
$$3y = x + z + u,$$
$$4z = x + y + u,$$
$$u = x - a.$$

Now, if x be added to the firſt, y to the ſecond, and z to the third, we ſhall get $(x + y + z + u) = 3x = 4y = 5z$; and from thence $z = \frac{3x}{5}$, and $y = \frac{3x}{4}$; which values being ſubſtituted in the firſt equation, we have $2x = \frac{3x}{4} + \frac{3x}{5} + u$, or $u = \frac{13x}{20}$; but, by the 4th equation, $u = x - a$; therefore $x - a = \frac{13x}{20}$, and $x = \frac{20a}{7} = 40$: conſequently $y\ (\frac{3x}{4}) = 30$, $z\ (\frac{3x}{5}) = 24$, and u, $(x - 14) = 26$; and the whole ſum $(x + y + z + u) = 120l.$

PROBLEM XXXIV.

To find four numbers, ſo that the firſt together with half the ſecond may be 357 *(a), the ſecond with* $\frac{1}{3}$ *of the third equal to* 476 *(b), the third with* $\frac{1}{4}$ *of the fourth equal to* 595 *(c), and the fourth with* $\frac{1}{5}$ *of the firſt equal to* 714 *(d).*

The required numbers being denoted by x, y, z and u, and the conditions of the queſtion expreſſed in algebraic terms, we have the four following equations:

$$x + \frac{y}{2} = a,$$

$$y + \frac{z}{3} = b,$$

$$z + \frac{u}{4} = c,$$

$$u + \frac{x}{5} = d.$$

From the firſt whereof we get $x = a - \frac{y}{2}$; and from the 4th, $x = 5d - 5u$; whence $a - \frac{y}{2} = 5d - 5u$, and $y = 2a - 10d + 10u$; but, by the ſecond, $y = b - \frac{z}{3}$; therefore $2a - 10d + 10u = b - \frac{z}{3}$, and $z = 3b - 6a + 30d - 30u$; but, by the third, $z = c - \frac{u}{4}$; whence $3b - 6a + 30d - 30u = c - \frac{u}{4}$, and $12b - 24a + 120d - 120u = 4c - u$; conſequently $u = \frac{12b - 24a + 120d - 4c}{119} = 676$; whence $z\ (c - \frac{u}{4}) = 426$, $y\ (= b - \frac{z}{3}) = 334$, and $x\ (= a - \frac{y}{2}) = 190$.

Otherwiſe.

Let the firſt of the required numbers be denoted by x (as above); then, the ſum of the firſt and ½ the ſecond, being given equal to a, it is manifeſt that ½ the ſecond muſt be equal to a *minus* the firſt, that is $= a - x$, and therefore the ſecond number $= 2a - 2x$: moreover, the ſum of the ſecond, and ⅓ of the third, being given $= b$; it is likewiſe evident, that ⅓ of the third muſt be equal to b *minus* the ſecond, that is $= b - 2a + 2x$, and conſequently the third number itſelf $= 3b - 6a + 6x$: In the ſame manner it will appear that ¼ of the fourth number $= c - 3b + 6a - 6x$; and conſequently the fourth number itſelf, $= 4c - 12b + 24a - 24x$: whence, *by the queſtion*, $4c - 12b + 24a - 24x + \frac{x}{5} = d$, and therefore $x = \frac{-5d + 20c - 60b + 120a}{119} = 190$; as above.

PROBLEM XXXV.

To divide the number 90 (a) into four ſuch parts, that, if the firſt be increaſed by 5 (b), the ſecond decreaſed by 4 (c), the third multiplied by 3 (d), and the fourth di-

vided by 2 *(c), the reſult, in each caſe, ſhall be exactly the ſame.*

Let x, y, z and u, be the parts required; then, *by the queſtion*, we ſhall have theſe equations, *viz.*

$$x + y + z + u = a, \text{ and}$$

$$x + b = y - c = dz = \frac{u}{e}.$$

Whence, by comparing dz with each of the three other equal values, ſucceſſively, $x = dz - b$, $y = dz + c$, and $u = dez$; all which, being ſubſtituted, for their equals, in the firſt equation, we thence get $dz - b + dz + c + z + dez = a$; whence $dez + 2dz + z = a + b - c$, and $z = \frac{a + b - c}{de + 2d + 1} = 7$. Therefore $x\ (= dz - b) = 16$; $y\ (= dz + c) = 25$; and $u\ (= dez) = 42$.

PROBLEM XXXVI.

If A *and* B *together, can perform a piece of work in* 8 *(a) days;* A *and* C *together in* 9 *(b) days, and* B *and* C *in* 10 *(c) days; how many days will it take each perſon, alone, to perform the ſame work?*

Let the three numbers ſought be repreſented by x, y and z, reſpectively: then it will be, as x (days) : a (days) :: 1, the whole work, to $\frac{a}{x}$, the part thereof performed by A in a days; and, as $y : a :: 1 : \frac{a}{y}$, the part performed by B, in the ſame time; whence, *by the queſtion*, $\frac{a}{x} + \frac{a}{y} = 1$ (the whole work). And, by proceeding in the very ſame manner, we ſhall have theſe two other equations, *viz.* $\frac{b}{x} + \frac{b}{z} = 1$, and $\frac{c}{y} + \frac{c}{z} = 1$: let the firſt of theſe three equations be divided by a, the ſecond by b, and the third by c, and you will then have $\frac{1}{x}$

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{a},$$

$$\frac{1}{x} + \frac{1}{z} = \frac{1}{b},$$

$$\frac{1}{y} + \frac{1}{z} = \frac{1}{c},$$

which added all together, and the ſum divided by 2, give $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{2a} + \frac{1}{2b} + \frac{1}{2c}$; from whence, each of the three laſt equations being ſucceſſively ſubtracted, we get

$$\frac{1}{z} = -\frac{1}{2a} + \frac{1}{2b} + \frac{1}{2c} = \frac{-bc + ac + ab}{2abc},$$

$$\frac{1}{y} = \frac{1}{2a} - \frac{1}{2b} + \frac{1}{2c} = \frac{bc - ac + ab}{2abc},$$

$$\frac{1}{x} = \frac{1}{2a} + \frac{1}{2b} - \frac{1}{2c} = \frac{bc + ac - ab}{2abc}.$$ Hence

$$z = \frac{2abc}{-bc + ac + ab} = \frac{1440}{-90 + 80 + 72} = 23\tfrac{7}{31},$$

$$y = \frac{2abc}{bc - ac + ab} = \frac{1440}{90 - 80 + 72} = 17\tfrac{23}{41},$$

$$x = \frac{2abc}{bc + ac - ab} = \frac{1440}{90 + 80 - 72} = 14\tfrac{34}{49}.$$

Otherwiſe.

Let the work performed by A in one day be denoted by x; then his work in a days will be ax, and in b days it will be bx; therefore the work of B in a days will be $1 - ax$; and that of C, in b days, $1 - bx$, by the conditions of the problem; whence it follows that the work of B, in one day, will be expreſſed by $\frac{1 - ax}{a}$, and that of C, in one day, by $\frac{1 - bx}{b}$; but the ſum of theſe two laſt is, *by the queſtion*, equal to $\frac{1}{c}$ part of the whole work, that is, $\frac{1}{a} + \frac{1}{b} - 2x = \frac{1}{c}$;

whence

whence $x = \frac{1}{2a} + \frac{1}{2b} - \frac{1}{2c} = \frac{bc + ac - ab}{2abc}$, equal to the work done by A in one day; by which divide 1 (the whole) and the quotient, $\frac{2abc}{bc + ac - ab}$, will give the required number of days in which he can finiſh the whole.

PROBLEM XXXVII.

To find three numbers, on theſe conditions, that a times the firſt, b times the ſecond, and c times the third, ſhall be equal to a given number p; that d times the firſt, e times the ſecond, and f times the third, ſhall be equal to another given number q; and that g times the firſt, h times the ſecond, and k times the third, ſhall be equal to a third given number r.

Let the three required numbers be denoted by x, y, and z, and then we ſhall have

$$\begin{aligned} ax + by + cz &= p, \\ dx + ey + fz &= q, \\ gx + hy + kz &= r. \end{aligned}$$

From d times the firſt of which ſubtract a times the ſecond, and from g times the firſt ſubtract a times the third, and you will have theſe two new equations,

$$\text{viz.} \left\{ \begin{array}{l} bdy - aey + cdz - afz = dp - aq, \\ bgy - ahy + cgz - akz = gp - ar; \end{array} \right.$$

or, which are the ſame,

$$\overline{bd - ae} \times y + \overline{cd - af} \times z = dp - aq,$$

$$\text{and, } \overline{bg - ah} \times y + \overline{cg - ak} \times z = gp - ar.$$

Multiply the firſt of theſe two equations by the coefficient of y in the ſecond, and *vice verſâ*, and let the laſt of the two products be ſubtracted from the former, and you will next have $\overline{cd - af} \times \overline{bg - ah} \times z - \overline{bd - ae} \times \overline{cg - ak} \times z = \overline{bg - ah} \times \overline{dp - aq} - \overline{bd - ae} \times \overline{gp - ar}$; and therefore $z = \frac{\overline{bg - ah} \times \overline{dp - aq} - \overline{bd - ae} \times \overline{gp - ar}}{\overline{cd - af} \times \overline{bg - ah} - \overline{bd - ae} \times \overline{cg - ak}}$; whence x and y may alſo be found.

Example. Let the given equations be

$$\begin{aligned} x + y + z &= 12, \\ 2x + 3y + 4z &= 38, \\ 3x + 6y + 10z &= 83; \end{aligned}$$

Or, which is the ſame thing, let $a = 1$, $b = 1$, $c = 1$, $p = 12$, $d = 2$, $e = 3$, $f = 4$, $q = 38$, $g = 3$, $h = 6$, $k = 10$, and $r = 83$: then theſe values being ſubſtituted above in that of z, it will become $\frac{\overline{3-6}\times\overline{24-38}-\overline{2-3}\times\overline{36-83}}{\overline{2-4}\times\overline{3-6}-\overline{2-3}\times\overline{3-10}}$ $= \frac{42 - 47}{6 - 7} = \frac{-5}{-1} = 5$; whence, alſo, we find $y\ (= \frac{dp - aq - \overline{cd - af} \times z}{bd - ae}) = \frac{24 - 38 - \overline{2 - 4} \times 5}{2 - 3}$ $= \frac{-4}{-1} = 4$, and $x\ (= \frac{p - by - cz}{a}) = \frac{12 - 4 - 5}{1}$ $= 3$.

Having exhibited a variety of examples of the uſe and application of Algebra, in the reſolution of problems producing ſimple equations, I ſhall now proceed to give ſome inſtances thereof in ſuch as riſe to quadratic equations; but, firſt of all, it will be neceſſary to premiſe ſomething, in general, with regard to theſe kinds of equations.

It has been already obſerved, that quadratic equations are ſuch wherein the higheſt power of the unknown quantity riſes to two dimenſions; of which there are two ſorts, *viz.* ſimple quadratics, and adfected ones. A ſimple quadratic equation, is that wherein the ſquare *only* of the unknown quantity is concerned, as $xx = ab$; but an adfected one is, when *both* the ſquare and its root are found involved in different terms of the ſame equation, as in the equation $x^2 + 2ax = bb$. The reſolution of the firſt of theſe is performed by, barely, extracting the ſquare root, on both ſides thereof: thus in the equation $x^2 = ab$, the value of x is given $= \sqrt{ab}$ (for if two quantities be equal, their ſquare roots muſt neceſſarily be equal). The method of ſolution when the equation is adfected, is likewiſe by extracting the ſquare root; but, firſt of all, ſo much is to be added to both ſides thereof as to make *that* where the unknown quantity is, a perfect ſquare; this is uſually called *completing the ſquare*, and is always done by taking half the coefficient of the ſingle power of the unknown quantity, in the ſecond term, and ſquaring it, and then adding that ſquare to both ſides of the equation. Thus, in the equation $xx + 2ax = bb$, the coefficient of x in the ſecond term being $2a$, its half will be a, which, ſquared and added to both ſides, gives $x^2 + 2ax + a^2 = b^2 + a^2$; whereof the former part is, now, a perfect ſquare. The ſquare being thus completed, its root is next to be extracted; in order to which, it is to be obſerved, that the root, on the left-hand ſide, where the unknown quantity ſtands, is compoſed of two terms, or members; whereof the former is always the ſquare root of the firſt term of the equation, and the latter the half of the coefficient of the ſecond term: thus, in the equation, $x^2 + 2ax + a^2 = b^2 + a^2$, before us, the ſquare root of the left-hand ſide, $x^2 + 2ax + a^2$, will be expreſſed by $x + a$ (for $\overline{x + a} \times \overline{x + a} = x^2 + 2ax + a^2$). Hence it is manifeſt that $x + a = \sqrt{b^2 + a^2}$, and therefore $x = \sqrt{b^2 + a^2} - a$; from which x is known. Theſe kinds of equations, it is alſo to be obſerved, are commonly divided into three forms, according to the different variations of the ſigns: thus $x^2 + 2ax = b^2$, is called an equation of the firſt form; $x^2 - 2ax = b^2$, one of the ſecond form; and $x^2 - 2ax = -b^2$ one of the third form; but the method of extracting the root, or finding the value of x, is the ſame in all three, except that, in the laſt of them, the root of the known part, on the right hand ſide, is to be expreſſed with the double ſign ± before it, x having two different affirmative values in this caſe. The reaſon of which, as well as of what has been ſaid in general, in relation to theſe kinds of equations, will plainly appear, by conſidering, that any ſquare, as $x^2 - 2ax + a^2$, raiſed from a binomial root, $x - a$ (or $a - x$) is compoſed of three members; whereof the firſt is the ſquare of the firſt term of the root; the ſecond, a rectangle of the firſt into twice the ſecond; and the third, the ſquare of the ſecond: from whence it is manifeſt, that, if the firſt and ſecond terms of the ſquare be given or expreſſed, not only the remaining term, but the root itſelf, will be found by the method above delivered.

But now, as to the ambiguity taken notice of in the third form, where $x^2 - 2ax = -b^2$, or $x^2 - 2ax + a^2 = a^2 - b^2$; the ſquare root of the left-hand ſide may be either $x - a$, or $a - x$ (for either of theſe, ſquared, produce the ſame quantity) therefore, in the former caſe, $x = a + \sqrt{a^2 - b^2}$, and in the latter, $x = a - \sqrt{a^2 - b^2}$; both which values anſwer the conditions of the equation. The ſame ambiguity would alſo take place in the other forms, were not the root (x) confined to a poſitive value.

When the higheſt power of the unknown quantity happens to be effected by a coefficient, the whole equation muſt be divided by that coefficient; and if the ſign of that power is negative, all the ſigns muſt be changed, before you ſet about to complete the ſquare.

All equations, whatever, in which there enter only two different dimenſions of the unknown quantity, whereof the index of the one is juſt double to that of the other, are ſolved like quadratics, by completing the ſquare: thus, the equation $x^4 + 2ax^2 = b$, by completing the ſquare will become $x^4 + 2ax^2 + a^2 = b + a^2$; whence, by extracting the root on both ſides, $x^2 + a = \sqrt{b + a^2}$; therefore $x^2 = \sqrt{b + a^2} - a$, and conſequently $x = \sqrt{\sqrt{b + a^2} - a}$.

Theſe things being premiſed, we now proceed on in the reſolution of Problems.

PROBLEM XXXVIII.

To find that number, to which 20 *being added, and from which* 10 *being ſubtracted, the ſquare of the ſum, added to twice the ſquare of the remainder, ſhall be* 17475.

Let the number ſought be denoted by x; then, by conditions of the queſtion, we ſhall have $\overline{x+20}|^2 + 2 \times \overline{x-10}|^2 = 17475$; that is, $x^2 + 40x + 400 + 2x^2 - 40x + 200 = 17475$; which, contracted, gives $3x^2 = 16875$. Hence $x^2 = 5625$; and conſequently, $x = \sqrt{5625} = 75$.

PROBLEM XXXIX.

To divide 100 *into two ſuch parts, that, if they be multiplied together the product ſhall be* 2100.

Let the exceſs of the greater part above (50) half the number given, be denoted by x; then $50 + x$ will be the greater part, and $50 - x$ the leſſer; therefore, *by the queſtion*, $\overline{50+x} \times \overline{50-x}$, or $2500 - x^2 = 2100$; whence $x^2 = 400$, and conſequently $x = \sqrt{400} = 20$; therefore $50 + x = 70 =$ the greater part, and $50 - x = 30 =$ the leſs.

PROBLEM XL.

What two numbers are thoſe, which are to one another in the ratio of 3 *(a) to* 5 *(b), and whoſe ſquares, added together, make* 1666 *(c)?*

Let the leſſer of the two required numbers be x; then, $a : b :: x : \frac{bx}{a} =$ the greater; therefore, *by the queſtion*, $x^2 + \frac{b^2x^2}{a^2} = c$; whence $a^2x^2 + b^2x^2 = a^2c$, and $x^2 = \frac{a^2c}{a^2+b^2}$; conſequently $x = \sqrt{\frac{a^2c}{a^2+b^2}} = a\sqrt{\frac{c}{a^2+b^2}} = 21 =$ leſſer number, and $\frac{bx}{a} = 35 =$ the greater.

PROBLEM XLI.

To find two numbers, whoſe difference is 8, *and product* 240.

If the leſſer number be denoted by x, the greater will be $x + 8$; and ſo, *by the queſtion*, we ſhall have $x^2 + 8x = 240$.

$= 240$. Now, by completing the ſquare, $x^2 + 8x + 16\ (= 240 + 16) = 256$; and, by extracting the root, $x + 4 = \sqrt{256} = 16$: whence $x = 16 - 4 = 12$; and $x + 8 = 20$; which are the two numbers that were to be found.

PROBLEM XLII.

To find two numbers whoſe difference ſhall be 12, and the ſum of their ſquares 1424.

Let the leſſer be x, and the greater will be $x + 12$; therefore, *by the problem*, $\overline{x + 12}|^2 + x^2 = 1424$, or $2x^2 + 24x + 144 = 1424$; this, ordered, gives $x^2 + 12x = 640$; which, by completing the ſquare, becomes $x^2 + 12x + 36\ (= 640 + 36) = 676$; whence, extracting the root on both ſides, we have $x + 6 = (\sqrt{676}) = 26$; therefore $x = 20$, and $x + 12 = 32$, are the two numbers required.

$$\text{For} \left\{ \begin{array}{l} 32 - 20 = 12, \\ 32^2 + 20^2 = 1424. \end{array} \right.$$

PROBLEM XLIII.

To divide 36 into three ſuch parts, that the ſecond may exceed the firſt by 4, and that the ſum of all their ſquares may be 464.

Let x be the firſt part, then the ſecond will be $x + 4$; and, the ſum of theſe two being taken from (36) the whole, we have $32 - 2x$, for the third, or remaining part; and ſo, *by the queſtion*, $x^2 + \overline{x + 4}|^2 + \overline{32 - 2x}|^2 = 464$, that is, $6x^2 - 120x + 1040 = 464$; whence $6x^2 - 120x = -576$, and $x^2 - 20x = -96$. Now, by completing the ſquare, $x^2 - 20x + 100\ (= 100 - 96) = 4$; and, by extracting the root, $x - 10 = \mp 2$. Therefore $x = 10 \mp 2$, that is, $x = 8$, or $x = 12$; ſo that 8, 12, and 16 are the three numbers required.

PROBLEM XLIV.

To divide the number 100 (a) into two ſuch parts, that their product and the difference of their ſquares, may be equal to each other.

Let the leſſer part be denoted by x, then the greater will be $a - x$, and we ſhall have $\overline{a - x} \times x = \overline{a - x}|^2 - x^2$, that is, $ax - x^2 = a^2 - 2ax$; whence $x^2 - 3ax = -a^2$; and, by completing the ſquare, $x^2 - 3ax + \frac{9a^2}{4} = (-a^2 + \frac{9a^2}{4})\ \frac{5a^2}{4}$; of which the root being extracted, there comes out $x - \frac{3a}{2} = \pm\sqrt{\frac{5aa}{4}}$, and therefore $x = \frac{3a}{2} \pm \sqrt{\frac{5aa}{4}}$. But x, by the nature of the problem, being leſs than a, the upper ſign (+) gives x too great; ſo that $x = \frac{3a}{2} - \sqrt{\frac{5aa}{4}} = 38,19658$ &c. muſt be the true value required.

PROBLEM XLV.

The ſum, and the ſum of the ſquares, of two numbers being given; to find the numbers.

Let half the ſum of the two numbers be denoted by a, half the ſum of their ſquares by b, and half the difference of the numbers by x; then will the numbers themſelves be repreſented by $a - x$, and $a + x$, and their ſquares by $a^2 - 2ax + x^2$, and $a^2 + 2ax + x^2$; and ſo we have $a^2 - 2ax + x^2 + a^2 + 2ax + x^2 = 2b$, *by the queſtion*. Which equation, contracted and divided by 2 gives $a^2 + x^2 = b$; whence $x^2 = b - a^2$, and conſequently $x = \sqrt{b - a^2}$. Therefore the numbers ſought are $a - \sqrt{b - a^2}$, and $a + \sqrt{b - a^2}$.

PROBLEM XLVI.

The ſum, and the ſum of the cubes of two numbers being given; to find the numbers.

Let the two numbers be expreſſed as in the preceding problem, and let the ſum of their cubes be denoted by c. Therefore will $\overline{a - x}|^3 + \overline{a + x}|^3 = c$, that is, by involution and reduction, $2a^3 + 6ax^2 = c$; whence

$6ax^2 = c - 2a^3$, $x^2 = \frac{c - 2a^3}{6a} = \frac{c}{6a} - \frac{a^2}{3}$, and $x = \sqrt{\frac{c}{6a} - \frac{a^2}{3}}$.

PROBLEM XLVII.

The ſum, and the ſum of the biquadrates (or 4th powers) of two numbers being given; to find the numbers.

The numbers being denoted as above, we ſhall here have $\overline{a - x}|^4 + \overline{a + x}|^4 = d$, that is, $2a^4 + 12a^2x^2 + 2x^4 = d$; from which, by tranſpoſition and diviſion, $x^4 + 6a^2x^2 = \frac{1}{2}d - a^4$; and, by completing the ſquare, $x^4 + 6a^2x^2 + 9a^4 = \frac{1}{2}d + 8a^4$; whence $x^2 + 3a^2 = \sqrt{\frac{1}{2}d + 8a^4}$; and, conſequently, $x = \sqrt{-3a^2 + \sqrt{\frac{1}{2}d + 8a^4}}$.

PROBLEM XLVIII.

The ſum, and the ſum of the 5th powers of two numbers being given; to find the numbers.

The notation in the preceding problems being ſtill retained, we ſhall have $2a^5 + 20a^3x^2 + 10ax^4 = e$; and therefore, $x^4 + 2a^2x^2 = \frac{e}{10a} - \frac{a^4}{5}$; and $x^2 + a^2 = \sqrt{\frac{e}{10a} + \frac{4a^4}{5}}$; whence $x = \sqrt{\sqrt{\frac{e}{10a} + \frac{4a^4}{5}} - a^2}$.

PROBLEM XLIX.

What two numbers are thoſe, whoſe product is 120 (a), and if the greater be increaſed by 8 (b), and the leſſer by 5 (c), the product of the two numbers thence ariſing ſhall be 300 (d)?

If the greater number be denoted by x, and the leſſer by y, we ſhall have

$xy = a$, and

$\overline{x + b} \times \overline{y + c} = d$, by the conditions of the queſtion. Subtract the firſt of theſe equations from the ſecond, and you will have $\overline{x + b} \times \overline{y + c} - xy = d - a$, that is, $cx + by + bc = d - a$; where both ſides being multiplied

plied by x (in order to exterminate y), we thence have $cx^2 + bxy + bcx = dx - ax$; but xy being $= a$, therefore is $bxy = ab$, and conſequently, by ſubſtituting this value in the laſt equation, $cx^2 + ab + bcx = dx - ax$; whence $cx^2 + bcx + ax - dx = -ab$, and therefore $x^2 + bx + \frac{ax}{c} - \frac{dx}{c} = -\frac{ab}{c}$; which, by making $f = \frac{d}{c} - \frac{a}{c} - b$ ($= 28$), will become $x^2 - fx = -\frac{ab}{c}$; hence $x^2 - fx + \frac{1}{4}f^2 = -\frac{ab}{c} + \frac{1}{4}f^2$, $x - \frac{1}{2}f = \pm \sqrt{\frac{1}{4}f^2 - \frac{ab}{c}}$, and $x = \frac{1}{2}f \pm \sqrt{\frac{1}{4}f^2 - \frac{ab}{c}} = 16$, or $= 12$; and conſequently y $(\frac{a}{x}) = 10$, or $= 7,5$.

$$\text{For} \begin{cases} 12 \times 10 = 120 \\ \overline{12 + 8} \times \overline{10 + 5} = 300. \end{cases}$$

$$\text{Alſo} \begin{cases} 16 \times 7,5 = 120 \\ \overline{16 + 8} \times \overline{7,5 + 5} = 300. \end{cases}$$

PROBLEM L.

To find two numbers, ſo that their ſum, their product, and the difference of their ſquares, may be all equal to one another.

The greater being denoted by x, and the leſſer by y, we have $x + y = xy$, and $x + y = x^2 - y^2$: the laſt of theſe equations, divided by $x + y$, gives $1 = x - y$; whence $x = 1 + y$; this value, ſubſtituted for x in the firſt equation, gives $1 + 2y = y + y^2$; therefore $y^2 - y = 1$, and $y = \frac{1}{2} + \sqrt{\frac{5}{4}}$; conſequently x $(1 + y) = \frac{3}{2} + \sqrt{\frac{5}{4}}$.

PROBLEM LI.

To divide the number 100 (a) *into two ſuch parts, that the ſum of their ſquare roots may be* 14 (b).

Let the greater part be x, and the leſſer will be $a - x$; therefore, *by the problem,* $\sqrt{x} + \sqrt{a - x} = b$; and, by ſquaring both ſides $x + 2\sqrt{ax - xx} + a - x = bb$; whence, by tranſpoſition and diviſion, $\sqrt{ax - xx} = \frac{bb - a}{2}$: therefore, by ſquaring again, $ax - xx = \frac{\overline{bb - a}^2}{4}$, or $x^2 - ax = (-\frac{\overline{bb - a}^2}{4}) = -\frac{b^4}{4} + \frac{b^2a}{2} - \frac{a^2}{4}$, and $x = \frac{a}{2} + \sqrt{\frac{2ab^2 - b^4}{4}} = \frac{a}{2} + \frac{b}{2}\sqrt{2a - b^2} = 64 =$ the greater part; whence $a - x = 36 =$ the leſſer part.

PROBLEM LII.

A grazier bought in as many ſheep as coſt him 60l. *out of which he reſerved* 15, *and ſold the remainder for* 54l. *and gained two ſhillings a head by them: the queſtion is, how many ſheep did he buy, and what did they coſt him a head?*

Let the number of ſheep be x; then if, 1200, the number of ſhillings which they all coſt, be divded by x, the quotient, $\frac{1200}{x}$, will, it is evident, be the number of ſhillings which they coſt him apiece; and ſo the number of ſhillings they were ſold at *per* head will be $\frac{1200}{x} + 2$, *by the queſtion*; and therefore this, multiplied by $x - 15$, the number of ſheep ſo ſold, will give $1200 + 2x - \frac{18000}{x} - 30$, equal to the whole number of ſhillings which they were all ſold for; that is, $1170 + 2x - \frac{18000}{x} = 1080$: hence we have $1170x + 2x^2 - 18000 = 1080x$, $2x^2 + 90x = 18000$, $x^2 + 45x = 9000$, and $x = \sqrt{9506.25} - 22.5 = 75$, the number of ſheep; and conſequently $\frac{1200}{75} = 16$ ſhillings, the price of each.

PRO-

PROBLEM LIII.

Two country-women, A *and* B, *betwixt them brought* 100 (*c*) *eggs to market; they both received the ſame ſum for their eggs, but* A (*who had the largeſt and beſt*) *ſays to* B, *Had I brought as many eggs as you I ſhould have received* 18 (*a*) *pence for them; but, replies* B, *Had I brought no more than you, I ſhould have received only* 8 (*b*) *pence for mine: the queſtion is, to find how many eggs each perſon had.*

If the number of eggs which A had be $= x$, the number of B's eggs will be $= c - x$; therefore, *by the problem*, it will be, $c - x : a :: x : \frac{ax}{c - x} =$ the number of pence which A received; and as $x : b :: c - x : \frac{b \times \overline{c - x}}{x} =$ the number of pence which B received: whence, again, *by the problem*, $\frac{ax}{c - x} = \frac{b \times \overline{c - x}}{x}$; and therefore $ax^2 = b \times \overline{c - x}|^2 = bc^2 - 2bcx + bx^2$; which equation, ordered, gives $x^2 + \frac{2bcx}{a - b} = \frac{bc^2}{a - b}$; from whence x comes out $\left(= \sqrt{\frac{bc^2}{a - b} + \frac{b^2c^2}{\overline{a - b}|^2}} - \frac{bc}{a - b}\right) = \frac{c\sqrt{ab} - bc}{a - b} = 40$. But the value of x may be otherwiſe, more readily, derived from the equation $ax^2 = b \times \overline{c - x}|^2$, without the trouble of completing the ſquare; for the ſquare root being extracted on both ſides thereof, we have $x\sqrt{a} = \overline{c - x} \times \sqrt{b}$; whence $x\sqrt{a} + x\sqrt{b} = c\sqrt{b}$, and conſequently $x = \frac{c\sqrt{b}}{\sqrt{a} + \sqrt{b}} = \frac{100\sqrt{8}}{\sqrt{18} + \sqrt{8}} = \frac{100\sqrt{4}}{\sqrt{9} + \sqrt{4}} = 40$, *as before.*

PROBLEM LIV.

One bought 120 pounds of pepper, and as many of ginger, and had one pound of ginger more for a crown than of pepper; and the whole price of the pepper exceeded that of the ginger by ſix crowns: how many pounds of pepper had he for a crown, and how many of ginger?

Let the number of pounds of pepper which he had for a crown be x, and the number of pounds of ginger will be $x + 1$; moreover, the whole price of the pepper will be $\frac{120}{x}$ crowns, and that of the ginger $\frac{120}{x+1}$; therefore, *by the queſtion*, $\frac{120}{x} - \frac{120}{x+1} = 6$; whence $120x + 120 - 120x = 6x^2 + 6x$, and therefore $x^2 + x = 20$; which, ſolved, gives $x = 4 =$ the pounds of pepper, and $x + 1 = 5 =$ thoſe of ginger.

PROBLEM LV.

To find three numbers in arithmetical progreſſion, whereof the ſum of the ſquares ſhall be 1232 (a), and the ſquare of the mean greater than the product of the two extremes by 16 (b).

Let the mean be denoted by x, and the common difference by y; then the numbers themſelves will be $x - y$, x, and $x + y$; and ſo, *by the problem*, we ſhall have theſe two equations,

$\overline{x-y}|^2 + x^2 + \overline{x+y}|^2 = a$, and

$x^2 = \overline{x-y} \times \overline{x+y} + b$: theſe contracted, become $3x^2 + 2y^2 = a$, and $x^2 = x^2 - y^2 + b$; from the latter whereof we get $y^2 = b = 16$; and conſequently $y = \sqrt{b} = 4$; which, ſubſtituted for y in the former, gives $3x^2 + 2b = a$; whence $x^2 = \frac{a-2b}{3}$, and therefore $x = \sqrt{\frac{a-2b}{3}} = 20$; ſo that the three required numbers are 16, 20, and 24.

$$\text{For} \begin{cases} 16^2 + 20^2 + 24^2 = 1232, \\ 20^2 - 16 \times 24 = 16. \end{cases}$$

PRO-

PROBLEM LVI.

To find two numbers whoſe difference ſhall be 10 (a), *and if* 600 (b) *be divided by each of them, the difference of the quotients ſhall alſo be equal to* 10 (a).

The leſſer number being repreſented by x, the greater will be repreſented by $x + a$; and therefore, *by the problem*, $\frac{b}{x} - \frac{b}{x + a} = a$; which, freed from fractions, gives $bx + ba - bx = ax^2 + a^2x$, that is, $ba = ax^2 + a^2x$; whence, dividing by a, and completing the ſquare, we have $x^2 + ax + \frac{1}{4}a^2 = b + \frac{1}{4}a^2$; therefore $x + \frac{1}{2}a = \sqrt{b + \frac{1}{4}a^2}$, and conſequently $x = \sqrt{b + \frac{1}{4}a^2} - \frac{1}{2}a = 20$ the leſſer number, whence $x + a = 30$, the greater number.

PROBLEM LVII.

To find two numbers whoſe ſum is 80 (a), *and, if they be divided alternately by each other, the ſum of the quotients ſhall be* $3\frac{1}{3}$ (b).

If one of the numbers be x, the other will be $a - x$, and we ſhall therefore have $\frac{x}{a - x} + \frac{a - x}{x} = b$: which equation, brought out of fractions, becomes $x^2 + a^2 - 2ax + x^2 = abx - bx^2$; and this, by tranſpoſition, gives $2x^2 + bx^2 - 2ax - abx = -a^2$, that is, $\overline{2 + b} \times x^2 - \overline{2 + b} \times ax = -a^2$; whereof both ſides being divided by $2 + b$, we have $x^2 - ax = -\frac{a^2}{2 + b}$; whence, by completing the ſquare, $x^2 - ax + \frac{a^2}{4} = \frac{a^2}{4} - \frac{a^2}{2 + b}$; hence $x - \frac{1}{2}a = \pm\sqrt{\frac{a^2}{4} - \frac{a^2}{a + b}}$, and $x = \frac{a}{2} \pm \sqrt{\frac{a^2}{4} - \frac{a^2}{2 + b}} = 60$, or $= 20$; which two, are the numbers that were to be found.

PROBLEM LVIII.

To divide the number 134 (a) *into three ſuch parts, that once the firſt, twice the ſecond, and three times the* *third, added together, may be* $= 278\ (b)$, *and that the ſum of the ſquares of all the three parts may be* $= 6036\ (c)$.

Let the three parts be denoted by x, y, and z, reſpectively; then, from the conditions of the problem, we ſhall have theſe three equations.

$$\begin{aligned} x + y + z &= a, \\ x + 2y + 3z &= b, \\ x^2 + y^2 + z^2 &= c. \end{aligned}$$

Let the firſt of theſe equations be ſubtracted from the ſecond, whence $y + 2z = b - a$, or $y = b - a - 2z$; alſo, if the double of the firſt be ſubtracted from the ſecond, there will come out $z - x = b - 2a$, or $x = z + 2a - b$: wherefore, if f be put $= b - a\ (= 144)$, $g = b - 2a\ (= 10)$, and for y and x, their equals $f - 2z$ and $z - g$, be ſubſtituted, our third equation, $x^2 + y^2 + z^2 = c$, will become $zz - 2gz + gg + ff - 4fz + 4zz + zz = c$; which, ordered, gives $z^2 - \frac{2f + g}{3} \times z = \frac{c - f^2 - g^2}{6}$; whence, by putting $b = \frac{2f + g}{3}\ (= \frac{298}{3})$, and completing the ſquare, &c. z is found $= \frac{b}{2} + \sqrt{\frac{c - f^2 - g^2}{6} + \frac{b^2}{4}}\ (= \frac{149}{3} + \frac{1}{3} =$ 50: therefore $y\ (= f - 2z) = 44$, and $x\ (= z - g) = 40$.

PROBLEM LIX.

A traveller ſets out from one city B, *to go to another* C, *at the ſame time as another traveller ſets out from* C *for* B; *they both travel uniformly, and in ſuch proportion, that the former, four hours after their meeting, arrives at* C, *and the latter at* B, *in nine hours after: now the queſtion is to find in how many hours each perſon performed the journey.*

Let D be the place of meeting, and put $a = 4$, $b = 9$, and $x =$ the number of hours they travel before they meet: then, the diſtances gone over, with the ſame uniform motion, being always to each other as the times in which they are deſcribed, we therefore have, BD : DC : : x (the time in which the firſt traveller goes the diſtance BD) : a (the time in which he goes the diſtance DC) : and, for the ſame reaſon, BD : DC : : b (the time in which the ſecond goes the diſtance BD) : x (the time in which he goes the diſtance DC) : wherefore, ſince it appears that x is to a in the ratio of BD to DC, and b to x in the ſame ratio, it follows that $x : a : : b : x$; whence $x^2 = ab$, and $x = \sqrt{ab}$ $(= 6)$; therefore $a + \sqrt{ab} = 10$, and $b + \sqrt{ab} = 15$, are the two numbers required.

PROBLEM LX.

There are four numbers in arithmetical progreſſion, whereof the product of the extremes is 3250 *(a); and that of the means* 3300 *(b): what are the numbers?*

Let the leſſer extreme be repreſented by y, and the common difference by x; then the four required numbers will be expreſſed by y, $y + x$, $y + 2x$, and $y + 3x$: therefore, *by the queſtion*, we have theſe two equations, *viz.*

$y \times \overline{y + 3x}$, or $y^2 + 3xy = a$, and

$\overline{y + x} \times \overline{y + 2x}$, or $y^2 + 3xy + 2x^2 = b$; whereof the former being taken from the latter, we get $2x^2 = b - a$: and from thence $x = \sqrt{\dfrac{b-a}{2}} = 5$. But, to find y from hence, we have given $y^2 + 3xy = a$ (by the firſt ſtep); therefore, by completing the ſquare, &c.

$y = \sqrt{a + \dfrac{9x^2}{4}} - \dfrac{3x}{2} = 50$: and ſo the four numbers are 50, 55, 60, and 65.

PROBLEM LXI.

The ſum (30), *and the ſum of the ſquares* (308) *of three numbers in arithmetical progreſſion being given; to find the numbers.*

Let the ſum of the numbers be repreſented by $3b$, the ſum of their ſquares by c, and the common difference by x: then, ſince the middle term, or number, from the nature of the progreſſion, is $= b$, or $\frac{1}{3}$ of the whole ſum, the leaſt term, it is evident, will be expreſſed by $b - x$, and the greateſt by $b + x$; and therefore, *by the queſtion*, we have this equation, $\overline{b - x}|^2 + b^2 + \overline{b + x}|^2 = c$; which, contracted, gives $3b^2 + 2x^2 = c$; whence $2x^2 = c - 3b^2$, and $x = \sqrt{\frac{c - 3b^2}{2}} = 2$. Therefore 8, 10, and 12, are the three numbers ſought.

PROBLEM LXII.

Having given the ſum (b), and the ſum of the ſquares (c) of any given number of terms in arithmetical progreſſion; to find the progreſſion.

Let the common difference be e, the firſt term $x + e$, and the number of terms n: then, *by the queſtion*, we ſhall have

$$\overline{x + e} + \overline{x + 2e} + \overline{x + 3e} \ldots\ldots \overline{x + ne} = b, \text{ and}$$

$$\overline{x + e}|^2 + \overline{x + 2e}|^2 + \overline{x + 3e}|^2 \ldots\ldots \overline{x + ne}|^2 = c.$$

But (*by Section* 10. *Theor*. 4.) the ſum of the firſt of theſe progreſſions is $nx + \frac{n . \overline{n + 1} . e}{2}$: And the ſum of the ſecond (as will be ſhewn further on) is $= nx^2 + n . \overline{n + 1} . xe + \frac{n . \overline{n + 1} . \overline{2n + 1} . e^2}{6}$: therefore our two equations will become

$$nx + \frac{n . \overline{n + 1} . e}{2} = b, \text{ and}$$

$$nx^2 + n . \overline{n + 1} . xe + \frac{n . \overline{n + 1} . \overline{2n + 1} . e^2}{6} = c.$$

Let the former whereof be ſquared, and the latter multiplied by n, and we ſhall thence have

$$n^2x^2 + n^2 . \overline{n + 1} . xe + \frac{n^2 . \overline{n + 1}|^2 . e^2}{4} = b^2, \text{ and}$$

$$n^2x^2 + n^2 . \overline{n + 1} . xe + \frac{n^2 . \overline{n + 1} . \overline{2n + 1} . e^2}{6} = nc:$$

let the firſt of theſe be ſubtracted from the ſecond, ſo ſhall $\frac{n^2 . \overline{n+1} . \overline{2n+1} . e^2}{6} - \frac{n^2 . \overline{n+1}|^2 . e^2}{4} = nc - b^2$.

But $\frac{n^2 . \overline{n+1} . \overline{2n+1}}{6} - \frac{n^2 . \overline{n+1}|^2}{4}$ is $= n^2 . \overline{n+1} \times \overline{\frac{2n+1}{6} - \frac{n+1}{4}} = n^2 . \overline{n+1} \times \frac{8n+4-6n-6}{24} = n^2 . \overline{n+1} . \frac{\overline{2n-2}}{24} = \frac{n^2 . \overline{n+1} . \overline{n-1}}{12} = \frac{n^2 . \overline{n^2-1}}{12}$:

Therefore $\frac{n^2 . \overline{n^2-1} . e^2}{12} = nc - b^2$, and $e = \sqrt{\frac{12nc - 12b^2}{n^2 \times \overline{n^2-1}}}$; whence x $(\frac{b}{n} - \frac{\overline{n+1} . e}{2})$ is known.

Example: Let the given number of terms be 6, their ſum 33, and the ſum of their ſquares 199; then, by writing theſe numbers, reſpectively, for n, b, and c, we ſhall have $e = 1$; whence $x = 2$, and the required numbers 3, 4, 5, 6, 7, and 8.

PROBLEM LXIII.

Two poſt-boys A *and* B *ſet out, at the ſame time, from two cities* 500 *miles aſunder, in order to meet each other;* A *rides* 60 *miles the firſt day,* 55 *the ſecond,* 50 *the third, and ſo on, decreaſing* 5 *miles every day: but* B *goes* 40 *miles the firſt day,* 45 *the ſecond,* 50 *the third,* &c. *increaſing five miles every day; now it is required to find in what number of days they will meet.*

In order to have a general ſolution to this problem, let the firſt day's diſtance of the poſt A be put $= m$, and the diſtance which he falls ſhort each day of the preceding $= d$; alſo the firſt day's diſtance of the poſt B $= p$, and the diſtance which he gains each day $= e$; and let x be the required number of days in which they meet: then the whole diſtance travelled by A will be expreſſed by the following arithmetical progreſſion, $m + \overline{m-d} + \overline{m-2d} + \overline{m-3d}$, &c. and that of B by

$p +$

$p + \overline{p+e} + \overline{p+2e} + \overline{p+3e}$, &c. where each progreſſion is to be continued to x terms. But the ſum of the firſt of theſe progreſſions (*by Sect.* 10. *Theor.* 4.) is = $mx - \frac{x \times \overline{x-1} \times d}{2}$, and that of the ſecond = $px + \frac{x \times \overline{x-1} \times e}{2}$: therefore theſe two laſt expreſſions, added together, muſt, by the conditions of the queſtion, be equal to 500 miles, the whole given diſtance; which we will call b, and then we ſhall have $\overline{p+m} \times x + \frac{x \times \overline{x-1} \times \overline{e-d}}{2} = b$, or $fx + \frac{gx \times \overline{x-1}}{2} = b$, by writing $f = p + m$, and $g = e - d$: which equation is reduced to $gx^2 - gx + 2fx = 2b$, or $x^2 - x + \frac{2fx}{g} = \frac{2b}{g}$; whence, by completing the ſquare, &c. x comes out $= \sqrt{\frac{2b}{g} + \overline{\frac{f}{g} - \frac{1}{2}}\Big|^{2}} - \frac{f}{g} + \frac{1}{2}$. But, in the particular caſe propoſed, the anſwer is more ſimple, and may be more eaſily derived from the firſt equation $\overline{p+m} \times x + \frac{x \times \overline{x-1} \times \overline{e-d}}{2} = b$; for, e being $= d$, $\frac{x \times \overline{x-1} \times \overline{e-d}}{2}$ will here entirely vaniſh out of the equation; and therefore x will be barely $= \frac{b}{p+m} = \frac{500}{40+60} = 5$. The ſame concluſion is alſo readily derived, without algebra, by the help of common arithmetic only: for ſeeing the ſum of the two diſtances travelled in the firſt day is 100 miles, and that the poſt B increaſes his diſtance, every day, by juſt as much as the poſt A decreaſes his, it is evident, that between them both, they muſt travel 100 miles every day; therefore, if 500 be divided by 100, the quotient 5 will be the number of days, in which they travel the whole 500 miles.

PROBLEM LXIV.

Two perſons, A *and* B, *ſet out together from the ſame place, and travel both the ſame way:* A *goes* 8 *miles the firſt day,* 12 *the ſecond,* 16 *the third, and ſo on, increaſing* 4 *miles every day: but* B *goes* 1 *mile the firſt day,* 4 *the ſecond,* 9 *the third, and ſo on, according to the ſquare of the number of days: the queſtion is, to find how many days each muſt travel before* B *comes up, again, with* A.

Let (4) the common difference of the progreſſion 8, 12, 16, &c. be put $= e$, and the firſt term thereof *minus* the ſaid common difference $= m$, and let the number of terms, or the days each perſon travels, be expreſſed by x: then the ſum of that progreſſion, or the number of miles which A travels will be $x \times m + \frac{x \times \overline{x+1} \times e}{2}$ (*by Sect.* 10. *Theor.* 4.) And (*by what follows hereafter*) the ſum of the progreſſion $1 + 4 + 9 \ldots . x^2$, or the diſtance travelled by B, will appear to be $\frac{x \times \overline{x+1} \times \overline{2x+1}}{6}$: therefore, *by the queſtion,* we have $\frac{x \times \overline{x+1} \times \overline{2x+1}}{6} = mx + \frac{x \times \overline{x+1} \times e}{2}$: which, divided by x and contracted, gives $\frac{2x^2 + 3x + 1}{6} = m + \frac{ex + e}{2}$; whence $x^2 + \frac{3x}{2} - \frac{3ex}{2} = 3m + \frac{3e}{2} - \frac{1}{2}$; and, by completing the ſquare, $x^2 + \frac{3x}{2} - \frac{3ex}{2} + \frac{9}{16} - \frac{18e}{16} + \frac{9e^2}{16}$ $\left(= 3m + \frac{3e}{2} - \frac{1}{2} + \frac{9}{16} - \frac{18e}{16} + \frac{9e^2}{16} = \frac{48m + 1 + 6e + 9e^2}{16}\right) = \frac{48m + \overline{1 + 3e}|^2}{16}$; whence

$x +$

$x + \frac{3}{4} - \frac{3e}{4} = \frac{\sqrt{48m + \overline{1 + 3e}|^2}}{4}$, and $x =$ $\frac{\sqrt{48m + \overline{1 + 3e}|^2} + 3e - 3}{4} = 7$, the number of days required.

PROBLEM LXV.

The ſum of the ſquares (a), and the continual product (b) of four numbers in arithmetical progreſſion being given; to find the numbers.

Let the common difference be denoted by $2x$, and the leſſer extreme by $y - 3x$; then, it is plain, the other three terms of the progreſſion will be expreſſed by $y - x$, $y + x$, and $y + 3x$ reſpectively; and ſo, *by the queſtion*, we have

$\overline{y - 3x}|^2 + \overline{y - x}|^2 + \overline{y + x}|^2 + \overline{y + 3x}|^2 = a$, and
$\overline{y - 3x} \times \overline{y - x} \times \overline{y + x} \times \overline{y + 3x} = b$,
that is, by reduction,

$$4y^2 + 20x^2 = a, \text{ and}$$
$$y^4 - 10y^2x^2 + 9x^4 = b;$$

from the former of which $y^2 = \frac{1}{4}a - 5x^2$: and therefore $y^4 = \frac{1}{16}a^2 - \frac{5}{2}ax^2 + 25x^4$: theſe values being ſubſtituted in the latter, we have $\frac{1}{16}a^2 - \frac{5}{2}ax^2 + 25x^4 - \frac{5}{2}ax^2 + 50x^4 + 9x^4 = b$, and therefore $x^4 - \frac{5ax^2}{84} = \frac{b}{84} - \frac{a^2}{16 \times 84}$; whence, by completing the ſquare $x^4 - \frac{5ax^2}{84} + \frac{25a^2}{4 \times 84 \times 84}$ $(= \frac{b}{84} + \frac{a^2}{84 \times 84}) = \frac{84b + a^2}{84 \times 84}$; therefore $x^2 - \frac{5a}{2 \times 84} = \frac{\pm\sqrt{84b + a^2}}{84}$, and $x = \sqrt{\frac{5a \pm 2\sqrt{84b + a^2}}{168}}$; whence y $(= \sqrt{\frac{1}{4}a - 5x^2})$ is alſo known.

PROBLEM LXVI.

The difference of the means (a), and the difference of the extremes (b) of four numbers in continued geometrical proportion being given; to find the numbers.

Let the ſum of the means be denoted by x; then the greater of them will be denoted by $\frac{x+a}{2}$, and the leſſer by $\frac{x-a}{2}$: whence, by the nature of proportionals, it will be $\frac{x+a}{2} : \frac{x-a}{2} :: \frac{x-a}{2} : \frac{\overline{x-a}|^2}{2x+2a}$, the leſſer extreme, and $\frac{x-a}{2} : \frac{x+a}{2} :: \frac{x+a}{2} : \frac{\overline{x+a}|^2}{2x-2a}$, the greater extreme: therefore, *by the problem*, we have $\frac{\overline{x+a}|^2}{2x-2a} - \frac{\overline{x-a}|^2}{2x+2a} = b$; and conſequently $\overline{x+a}|^3 - \overline{x-a}|^3 = 2b \times \overline{x-a} \times \overline{x+a}$, that is, $6x^2a + 2a^3 = 2b \times \overline{x^2-a^2}$; whence $x^2 = \frac{ba^2+a^3}{b-3a}$, and conſequently $x = a\sqrt{\frac{b+a}{b-3a}}$.

PROBLEM LXVII.

The ſum, and the ſum of the ſquares of three numbers in geometrical proportion being given; to find the numbers.

Let the ſum of the three numbers be denoted by a, and the ſum of their ſquares by b, and let the numbers themſelves be denoted by x, y, and z: then we ſhall have

$$x + y + z = a,$$
$$x^2 + y^2 + z^2 = b,$$
$$\text{and } xz = y^2.$$

Tranſpoſe y in the firſt equation, and ſquare both ſides, ſo ſhall $x^2 + 2xz + z^2 = a^2 - 2ay + y^2$; from whence ſubtracting the ſecond equation, we have $2xz - y^2 = a^2 - 2ay + y^2 - b$: but, by the third, $2xz = 2y^2$; therefore $y^2 = a^2 - 2ay + y^2 - b$; and conſequently $y = \frac{a^2 - b}{2a} = \frac{a}{2} - \frac{b}{2a}$. Now, to find x and z, y may be looked upon as known; and ſo, by the ſecond equation, we have given $x^2 + z^2 = b - y^2$; from which ſubtracting $2xz = 2y^2$, there ariſes $x^2 - 2xz + z^2 = b - 3y^2$; where, the ſquare root being extracted, we have $x - z = \sqrt{b - 3y^2}$: but, by the firſt equation, we have $x + z = a - y$; whence, by adding and ſubtracting theſe laſt equations, there reſults $2x = a - y + \sqrt{b - 3yy}$, and $2z = a - y - \sqrt{b - 3yy}$.

PROBLEM LXVIII.

The ſum (s), *and the product* (p) *of any two numbers being given; to find the ſum of the ſquares, cubes, biquadrates,* &c. *of thoſe numbers.*

If the two numbers be denoted by x and y; then will

$$\left.\begin{array}{r} x + y = s \\ \text{and } xy = p \end{array}\right\} \textit{by the problem.}$$

The former of which, ſquared, gives $xx + 2xy + yy$; from whence ſubtracting the double of the latter, we have $x^2 + y^2 = s^2 - 2p$, *the ſum of the ſquares.*

Let this equation be multiplied by $x + y = s$; ſo ſhall $x^3 + xy \times \overline{x + y} + y^3 = s^3 - 2sp$, that is, $x^3 + p \times s + y^3 = s^3 - 2sp$ (becauſe $xy = p$, and $x + y = s$), and therefore $x^3 + y^3 = s^3 - 3sp$, *the ſum of the cubes.*

Multiply, again, by $x + y = s$, then will $x^4 + xy \times \overline{x^2 + y^2} + y^4 = s^4 - 3s^2p$, or $x^4 + p \times \overline{s^2 - 2p} + y^4 = s^4 - 3s^2p$ (becauſe $x^2 + y^2 = s^2 - 2p$). Conſequently $x^4 + y^4 = s^4 - 4s^2p + 2p^2$, *the ſum of the biquadrates.*

Hence the law of continuation is manifeſt, being ſuch, that the ſum of the next ſuperior powers will be *always* obtained by multiplying the ſum of the powers laſt found by s, and ſubtracting, from the product, the ſum of the preceding ones multiplied by p. And the ſum of the nth powers, expreſſed in a general manner, will be $s^n - ns^{n-2}p + n \,.\, \frac{n-3}{2} \,.\, s^{n-4}p^2 - n \,.\, \frac{n-4}{2}$.

$\frac{n-5}{3} \cdot s^{n-6} p^3 + n \cdot \frac{n-5}{2} \cdot \frac{n-6}{3} \cdot \frac{n-7}{4} \cdot s^{n-8} p^4$, *&c.*

PROBLEM LXIX.

The ſum of the ſquares (a), and the exceſs (b) of the product above the ſum of two numbers being given; to find the numbers.

Let the ſum of the numbers be denoted by s, and their product by r: then the ſum of the ſquares will be $s^2 - 2r$ (*by the laſt problem*), and we ſhall have $r - s = b$, and $s^2 - 2r = a$, whence, by adding the double of the former equation to the latter, $s^2 - 2s = a + 2b$; and conſequently $s = \sqrt{a + 2b + 1} + 1$. From which r ($= b + s$) is likewiſe known; and from thence the numbers themſelves.

PROBLEM LXX.

The ſum (a), and the ſum of the ſquares (b) of four numbers, in geometrical progreſſion, being given; to find the numbers.

If x and y be taken to denote the two middle numbers, the two extreme ones, by the nature of progreſſionals, will be truly repreſented by $\frac{x^2}{y}$ and $\frac{y^2}{x}$.

Put the ſum of the two means $= s$, and their rectangle $= r$; ſo ſhall the ſum of the two extremes ($\frac{xx}{y} + \frac{yy}{x}$) be $= a - s$, and their rectangle alſo $= r$ (by the nature of the queſtion). But (*by Problem* 68) the ſum of the ſquares of any two numbers whoſe ſum is s, and rectangle r, will be $= ss - 2r$; and (for the very ſame reaſon) the ſum of the ſquares of our other two numbers (whoſe ſum is $a - s$, and rectangle r) will be $= \overline{a-s}|^2 - 2r$. Therefore, by adding theſe aggregates of the ſquares of the means and extremes together, we get this equation, *viz.* $s^2 + \overline{a-s}|^2 - 4r = b$.

Moreover, from the equation, $\frac{xx}{y} + \frac{yy}{x} = a - s$, we get $x^3 + y^3 = xy \times \overline{a - s} = r \times \overline{a - s}$: but (*by the ſame Prob. juſt now quoted*) $x^3 + y^3 = s^3 - 3sr$; therefore $s^3 - 3sr = ar - sr$, or $r = \frac{s^3}{2s + a}$; which value being ſubſtituted for r, in the preceding equation, we have $s^2 + \overline{a - s}|^2 - \frac{4s^3}{2s + a} = b$. This, ſolved, gives $s = \sqrt{\frac{aa - b}{2} + \frac{bb}{4aa}} - \frac{b}{2a}$: whence every thing elſe is readily found.

PROBLEM LXXI.

The ſum (a) *and the ſum of the ſquares* (b) *of five numbers, in geometrical progreſſion, being given, to find the numbers.*

Let the three middle numbers be denoted by x, y, and z: then the two extreme ones will be $\frac{xx}{y}$ and $\frac{zz}{y}$; and therefore we ſhall have

$$\left.\begin{array}{l} \frac{xx}{y} + x + y + z + \frac{zz}{y} = a, \\ \frac{x^4}{y^2} + x^2 + y^2 + z^2 + \frac{z^4}{y^2} = b, \end{array}\right\} \textit{by the queſtion.}$$

Put $x + z = u$; then, by the firſt equation, $\frac{xx}{y} + \frac{zz}{y} = a - u - y$. Wherefore, ſeeing the ſum of the two extremes is expreſſed by $a - u - y$, and their rectangle by y^2 (*ſee Theor.* 7. *Sect.* 10), the ſum of their ſquares will (*by Problem* 68) be $= \overline{a - u - y}|^2 - 2y^2$: and, in the very ſame manner, the ſum of the ſquares of the two terms (x and z) adjacent to the middle one (y) will be $= u^2 - 2y^2$. Whence, by ſubſtituting theſe values, our equations become $\frac{u^2 - 2y^2}{y} + u + y = a$, and

$a -$

$\overline{a-u-y}|^2 - 2y^2 + u^2 - 2y^2 + y^2 = b$; which, by reduction, are changed to

$$aa - 2au - 2ay + 2uu + 2uy - 2yy = b,$$
$$\text{and } ay - uu - uy + yy = 0.$$

To the former of which add the double of the latter, ſo ſhall $aa - 2au = b$; and therefore $u = \frac{a}{2} - \frac{b}{2a}$. From whence, and $yy + \overline{a-u} \times y = uu$, the value of y $\left(= \sqrt{uu + \frac{\overline{a-u}|^2}{4}} - \frac{a-u}{2}\right)$ is likewiſe given.

PROBLEM LXXII.

The ſum (a), the ſum of the ſquares (b), and the ſum of the cubes (c), of any four numbers in geometrical proportion being given; to find the numbers.

Let half the ſum of the two means be x, and half their difference y; alſo let half the ſum of the two extremes be z, and half their difference v, and then the numbers themſelves will be expreſſed thus, $z - v$, $x - y$, $x + y$, $z + v$: whence, by the conditions of the problem, we have

$$\overline{z-v} + \overline{x-y} + \overline{x+y} + \overline{z+v} = a,$$
$$\overline{z-v}|^2 + \overline{x-y}|^2 + \overline{x+y}|^2 + \overline{z+v}|^2 = b,$$
$$\overline{z-v}|^3 + \overline{x-y}|^3 + \overline{x+y}|^3 + \overline{z+v}|^3 = c,$$
$$\overline{z-v} \times \overline{z+v} = \overline{x-y} \times \overline{x+y}$$

(Theor. 1, p. 72); which, contracted, are,

$$2z + 2x = a,$$
$$2z^2 + 2v^2 + 2x^2 + 2y^2 = b,$$
$$2z^3 + 6zv^2 + 2x^3 + 6xy^2 = c,$$
$$z^2 - v^2 = x^2 - y^2.$$

Let $x^2 - z^2 + v^2$, the value of y^2, in the laſt of theſe equations, be ſubſtituted inſtead of y^2, in the two preceding ones, and we ſhall have

$$2z^2 + 2v^2 + 2x^2 + 2x^2 - 2z^2 + 2v^2 = b, \text{ and}$$
$$2z^3 + 6zv^2 + 2x^3 + 6x^3 - 6xz^2 + 6xv^2 = c;$$

which, abbreviated, become

$$4x^2 + 4v^2 = b, \text{ and}$$
$$2z^3 + 8x^3 - 6xz^2 + \overline{6x + 6z} \times v^2 = c.$$

Let $\frac{1}{4}b - x^2$, the value of v^2, in the former of theſe equations, be ſubſtituted, for its equal, in the latter, and we ſhall next have $2z^3 + 8x^3 - 6xz^2 + \overline{6x + 6z} \times \overline{\frac{1}{4}b - x^2} = c$; moreover, if for z, in the laſt equation, its equal $\frac{1}{2}a - x$ be ſubſtituted, there will come out $2 \times \overline{\frac{1}{2}a - x}|^3 + 8x^3 - 6x \times \overline{\frac{1}{2}a - x}|^2 + 3a \times \overline{\frac{1}{4}b - xx} = c$; that is, $6ax^2 - 3a^2x + \frac{a^3}{4} + \frac{3ab}{4} = c$; therefore $x^2 - \frac{ax}{2} = \frac{c}{6a} - \frac{b}{8} - \frac{a^2}{24}$; and conſequently $x = \frac{a}{4} - \sqrt{\frac{c}{6a} - \frac{b}{8} + \frac{a^2}{48}}$, whence, z, v, and y, are likewiſe known.

The ſame otherwiſe.

Let the ſum of the two means $= s$, and their rectangle $= r$; ſo ſhall the ſum of the two extremes $= a - s$, and their rectangle alſo $= r$ *(by the queſtion)*: from whence, and *Prob.* 68, it is evident, that the ſum of the ſquares of the means will be $= s^2 - 2r$, and the ſum of the ſquares of the extremes $= \overline{a - s}|^2 - 2r$; alſo, that the ſum of the cubes of the means will be $= s^3 - 3rs$, and that of the extremes $= \overline{a - s}|^3 - 3r \times \overline{a - s}$: by means whereof, and the conditions of the problem, we have given the two following equations.

viz. $s^2 + \overline{a - s}|^2 - 4r = b$, or, $2s^2 - 2as - 4r = b - aa$;
and $s^3 + \overline{a - s}|^3 - 3ra = c$, or, $3as^2 - 3a^2s - 3ar = c - a^3$:
divide the former by 2, and the latter by $3a$, and then ſubtract the one from the other, ſo ſhall $r = \frac{aa}{6} - \frac{b}{2} + \frac{c}{3a}$; whence the value of s $(= \frac{a}{2} - \sqrt{\frac{b - aa}{2} + 2r + \frac{aa}{4}}$, *by the firſt equation)* is alſo given,

given, being (when ſubſtitution is made) $= \frac{a}{2} - \sqrt{\frac{aa}{12} - \frac{b}{2} + \frac{2c}{3a}}$.

PROBLEM LXXIII.

Having given the ſum (a), and the ſum of the ſquares (b), of any number of quantities in geometrical progreſſion; to determine the progreſſion.

Let the firſt term be denoted by x, the common ratio by z, and the given number of terms by n: then, by the conditions of the problem, we ſhall have

$$x + xz + xz^2 + xz^3 + xz^4 \ldots + xz^{n-1} = a,$$
$$x^2 + x^2z^2 + x^2z^4 + x^2z^6 + x^2z^8 \ldots + x^2z^{2n-2} = b.$$

Multiply the firſt equation by $1 - z$, and the ſecond by $1 - z^2$; ſo ſhall

$$x - xz^n = a \times \overline{1 - z}, \text{ and}$$
$$x^2 - x^2z^{2n} = b \times \overline{1 - z^2}.$$

Divide the latter of *theſe* by the former; whence will be had $x + xz^n = \frac{b}{a} \times \overline{1 + z}$: let this equation and the firſt be now multiplied croſs-wiſe, into each other, in order to exterminate x; ſo ſhall $a \times \overline{1 + z^n} = \frac{b}{a} \times \overline{1 + z} \times \overline{1 + z + z^2 + z^3 \ldots z^{n-1}}$.

If n be an even number, put $2m = n$; then our laſt equation, when multiplication by $1 + z$ is actually made, will ſtand thus, $\frac{aa}{b} \times \overline{1 + z^{2m}} = 1 + 2z + 2z^2 \ldots\ldots + 2z^{2m-2} + 2z^{2m-1} + z^{2m}$; which, divided by z^m, becomes $\frac{aa}{b} + \overline{\frac{1}{z^m} + z^m} = \frac{1}{z^m} + \frac{2}{z^{m-1}} + \frac{2}{z^{m-2}} \ldots\ldots + \frac{2}{z^2} + \frac{2}{z} + 2 + 2z + 2z^2 \ldots\ldots + 2z^{m-2}$

$+ 2z^{m-1} + z^m$. Let s be now put $(= \frac{1}{z} + z) =$ the ſum of the halves of the two terms of the ſeries adjacent to (2) the middle one; then, the rectangle of theſe quantities being 1, the ſum of their ſquares (or half the ſum of the two terms of the ſeries next to thoſe) will be $= s^2 - 2$ (*by Problem* 68); and the ſum $(\frac{1}{z^3} + z^3)$ of half the two next terms to theſe laſt $= s^3 - 3s$, &c. &c.

Hence, by making $d = \frac{aa}{2b} - \frac{1}{2}$ and putting the value of $\frac{1}{z^m} + z^m$ (*as expreſſed in the ſaid problem* 68) $= Q$, and then ſubſtituting above, *&c.* our equation becomes $dQ = 1 + s + \overline{s^2 - 2} + \overline{s^3 - 3s} + \overline{s^4 - 4s^2 + 2}$, &c. *continued to m terms*; whence the value of s may be determined.

Thus, let n, the number of terms given, be four; then m being $= 2$, $Q\ (= \frac{1}{z^2} + z^2)$ will be $s^2 - 2$; and our equation will, *here*, be $d \times \overline{s^2 - 2} = 1 + s$. If n be $= 6$, $Q\ (= \frac{1}{z^3} + z^3)$ will be $= s^3 - 3s$; and we ſhall have $d \times \overline{s^3 - 3s} = 1 + s + \overline{s^2 - 2} = s^2 + s - 1$; and ſo in other caſes, where n is an even number.

If n be an odd number, put $2m = n - 1$; and let both ſides of the equation

$$a \times \overline{1 + z^n} = \frac{b}{a} \times \overline{1 + z} \times \overline{1 + z + z^2 \ldots z^{n-1}}$$

be divided by $1 + z$; ſo ſhall

$$a \times \overline{1 - z + z^2 - z^3 \ldots - z^{n-2} + z^{n-1}} = \frac{b}{a} \times \overline{1 + z + z^2 \ldots + z^{n-1}}$$

(becauſe $\overline{1 + z} + \overline{1 - z + z^2 - z^3 + z^4 \ldots - z^{n-2} + z^{n-1}}$

$= 1$

$$= \left\{ \begin{array}{l} 1 - z + z^2 - z^3 + z^4 \ldots - z^{n-2} + z^{n-1} \quad * \\ * + z - z^2 + z^3 - z^4 \ldots + z^{n-2} - z^{n-1} + z^n \end{array} \right\} =$$

$1 + z^n)$: whence, by tranſpoſition, and ſubſtituting m,

$$\overline{a - \frac{b}{a}} \times \overline{1 + z^2 + z^4 \ldots + z^{2m}} = \overline{a + \frac{b}{a}} \times \overline{z + z^3 + z^5 \ldots z^{2m-1}};$$

put $\frac{aa + b}{aa - b} = c$ and let the whole equation be divided by $\overline{a - \frac{b}{a}} \times z^m$; then will

$$\frac{1}{z^m} + \frac{1}{z^{m-2}} + \frac{1}{z^{m-4}} \ldots + z^{m-4} + z^{m-2} + z^m = c \times \overline{\frac{1}{z^{m-1}} + \frac{1}{z^{m-3}} \ldots + z^{m-3} + z^{m-1}}.$$

Now, if m be an even number, the powers of z in the former part of the equation will be the even ones, aud thoſe in the latter the odd ones; but if m be an odd number, then, *vice verſâ.*

In the firſt caſe our equation may be wrote thus,

$$\frac{1}{z^m} + \frac{1}{z^{m-2}} \ldots + \frac{1}{z^4} + \frac{1}{z^2} + 1 + z^2 + z^4 \ldots + z^{m-2} + z^m = c \times \overline{\frac{1}{z^{m-1}} + \frac{1}{z^{m-3}} \ldots + \frac{1}{z^3} + \frac{1}{z} + z + z^3 \ldots z^{m-3} + z^{m-1}}.$$

Where, ſince $\frac{1}{z} + z = s$, $\frac{1}{z^2} + z^2 = s^2 - 2$, $\frac{1}{z^3} + z^3 = s^3 - 3s$, $\frac{1}{z^4} + z^4 = s^4 - 4s^2 + 2$, &c. we ſhall, by ſubſtituting theſe values in each ſeries (proceeding from the middle both ways) have $1 + \overline{s^2 - 2} + \overline{s^4 - 4s^2 + 2} + \text{\&c.} = c$ into $s + \overline{s^3 - 3s} + \text{\&c.}$

But, in the ſecond caſe, where m is an odd number, and the even powers of z come into the ſecond ſeries, we ſhall, by the very ſame method, have $s + \overline{s^3 - 3s} + \overline{s^5 - 5s^3 + 5s} + \text{\&c.} = c$ into $1 + \overline{s^2 - 2s} + \overline{s^4 - 4s^2 + 2} + \text{\&c.}$

In both which caſes the terms are to be ſo far continued, that the exponent of s, in the higheſt of them, may be $= \frac{n-1}{2}$. Thus, if n, the given number of terms, be 3, then m $(\frac{n-1}{2})$ being $= 1$, the equation belongs to *caſe* 2, and will be $s = c$, *barely*. If $n = 5$, then $m = 2$; and therefore $1 + s^2 - 2 = cs$, or $s^2 - 1 = cs$, *by caſe* 1. If n be 7, m will be 3; and ſo $s + s^3 - 3s = c \times \overline{1 + s^2 - 2}$, or $s^3 - 2s = c \times \overline{s^2 - 1}$, *by caſe* 2. Laſtly, if $n = 9$, then $m = 4$, and therefore $1 + \overline{s^2 - 2} + \overline{s^4 - 4s^2 + 2} = c \times \overline{s + s^3 - 3s}$, or $s^4 - 3s^2 + 1 = c \times \overline{s^3 - 2s}$, *by caſe* 1.

PROBLEM LXXIV.

Having given the ſum (a), and the ſum of the cubes (b), of any number of terms in geometrical progreſſion; to determine the progreſſion.

By retaining the notation in the laſt problem, and proceeding in the ſame manner, we here have

$$a = x + xz + xz^2 \ldots + xz^{n-1} = \frac{xz^n - x}{z - 1}, \text{ and}$$

$$b = x^3 + x^3z^3 + x^3z^6 \ldots + x^3z^{3n-3} = \frac{x^3z^{3n} - x^3}{z^3 - 1}$$ (by *Theorem* 8. *Sect.* 10).

Divide the laſt of theſe equations by the former, ſo ſhall $\frac{b}{a} = x^2 \times \frac{\overline{z-1} \times \overline{z^{3n}-1}}{\overline{z^3-1} \times \overline{z^n-1}} = x^2 \times \frac{z^{2n} + z^n + 1}{z^2 + z + 1}$ (becauſe $\frac{z^3 - 1}{z - 1} = z^2 + z + 1$, and $\frac{z^{3n} - 1}{z^n - 1} = z^{2n} + z^n + 1$). Let this equation, and the ſquare of the firſt, $a^2 = x^2 \times \frac{z^{2n} - 2z^n + 1}{z^2 - 2z + 1}$, be now multiplied, croſs-wiſe, in order to exterminate x; whence will be had $\frac{b}{a} \times \frac{z^{2n} - 2z^n + 1}{z^2 - 2z + 1} = a^2 \times \frac{z^{2n} + z^n + 1}{z^2 + z + 1}$: which, the numerators being divided by z^n, and the denominators by z, will ſtand thus,

$$b \times \frac{z^n - 2 + \frac{1}{z^n}}{z - 2 + \frac{1}{z}} = a^3 \times \frac{z^n + 1 + \frac{1}{z^n}}{z + 1 + \frac{1}{z}}.$$

Put (as before) the ſum of z and $\frac{1}{z} = s$; then, their rectangle being 1, the ſum of their *n*th powers $(z^n + \frac{1}{z^n})$ will be had in terms of s (*from Problem* 68), which ſum let be denoted by S; ſo ſhall our equation become $b \times \frac{S - 2}{s - 2} = a^3 \times \frac{S + 1}{s + 1}$: whence the value of s may, in any caſe, be determined.

Thus if (n) the given number of terms be 3; then S (the ſum of the cubes of z and $\frac{1}{z}$) being $= s^3 - 3s$, we have $b \times \frac{s^3 - 3s - 2}{s - 2} = a^3 \times \frac{s^3 - 3s + 1}{s + 1}$; that is, by diviſion, $b \times \overline{s^2 + 2s + 1} = a^3 \times \frac{s^3 - 3s + 1}{s + 1}$.

If the number of terms be 4; then will $S = s^4 - 4s^2 + 2$; and therefore $b \times \frac{s^4 - 4s^2}{s - 2} = a^3 \times \frac{s^4 - 4s^2 + 3}{s + 1}$; which, by an actual diviſion of the numerators, is reduced to $b \times \overline{s^3 + 2s^2} = a^3 \times \overline{s^3 - s^2 - 3s + 3}$.

Again, taking $n = 5$, we have $S = s^5 - 5s^3 + 5s$; and therefore $b \times \frac{s^5 - 5s^3 + 5s - 2}{s - 2} = a^3 \times \frac{s^5 - 5s^3 + 5s + 1}{s + 1}$ which, by diviſion, is reduced to $b \times \overline{s^4 + 2s^3 - s^2 - 2s + 1} = a^3 \times \overline{s^4 - s^3 - 4s^2 + 4s + 1}$; and ſo of others; where it may be obſerved, that the values of $S - 2$, and $S + 1$, will be always diviſible by their reſpective denominators; except the latter, when n is either 3, or a multiple of 3.

PROBLEM LXXV.

The ſum of any rank of quantities ($a + b + c + d + e +$ &c.) *being given* $= P$, *the ſum of all their rectangles* ($ab + ac + ad$ &c. $+ bc + bd$ &c. $+ cd$ &c.) $= Q$, *the ſum of all their ſolids* ($abc + abd + abe$ &c. $+ acd + ace$ &c. $+ bcd$ &c.) $= R$, &c. &c. *it is propoſed to determine the ſum of the ſquares, cubes, biquadrates,* &c. *of thoſe quantities.*

Put
- $p = b + c + d$ &c. = ſum of all the quantities after the firſt (a),
- $q = bc + bd + be$ &c. $+ cd + ce$ &c. = the ſum of their rectangles,
- $r = bcd + bce$ &c. $+ cde$ &c. = the ſum of their ſolids.
- &c. &c.

Then will
$$\begin{aligned} P &= a + p, \\ Q &= pa + q, \\ R &= qa + r, \\ S &= ra + s, \\ T &= sa + t, \text{ \&c.} \end{aligned}$$

By ſquaring the firſt of which equations, we have $P^2 = a^2 + 2ap + p^2$; from whence the double of the ſecond being ſubtracted (in order to exterminate $2ap$), there reſults $P^2 - 2Q = a^2 + p^2 - 2q$. Where $P^2 - 2Q$ expreſſes the true ſum of all the propoſed ſquares $a^2 + b^2 + c^2 + d^2$ &c.; becauſe, all the quantities $a, b, c, d,$ &c. being concerned exactly alike in the original, or given equations, they muſt neceſſarily be alike concerned in the concluſions thence derived; ſo that if ſubſtitution for p and q were to be actually made in the equation $P^2 - 2Q = a^2 + p^2 - 2q$, here brought out, it is evident that no other dimenſions of $b, c, d, e,$ &c. beſides the ſquares, can remain therein, as no dimenſions of a, beſides its ſquare, has place in this equation.

In order to find the ſum of all the cubes, put $A\ (= P) = a + p =$ ſum of the roots, and $B\ (= P^2 - 2Q) = a^2 + p^2 - 2q =$ ſum of the ſquares; then, by multiplying the two equations together, we have $PB = a^3 + pa^2 + p^2a - 2qa + p^3 - 2pq$.

From whence (to exterminate pa^2 the next inferior power of a after the higheſt, a^3) let $QA = pa^2 + p^2a + qa + pq$ (the product of the equations Q and A) be ſubducted; and there will remain $PB - QA = a^3 - 3qa + p^3 - 3pq$. To this laſt equation (in order to take away the next inferior power of a) add three times the equation $R = qa + r$, ſo ſhall $PB - QA + 3R = a^3 + p^3 - 3pq + 3r$. From whence it is evident that $PB - QA + 3R$ muſt be the required ſum of all the cubes $a^3 + b^3 + c^3 + d^3$ &c. for reaſons already ſpecified with reſpect to the preceding caſe.

To determine the ſum of the biquadrates, put $C = a^3 + p^3 - 3pq + 3r =$ the ſum of all the cubes; then multiplying by the equation $P = a + p$ (*as before*), we get $PC = a^4 + pa^3 + p^3a - 3pqa + 3ra + p^4 - 3p^2q + 3pr$. From which (to exterminate pa^3) ſubtract $QB = pa^3 + p^3a - 2pqa + qa^2 + p^2q - 2q^2$ (the product of the equations Q and B); ſo ſhall $PC - QB = a^4 - qa^2 - pqa + 3ra + p^4 - 4p^2q + 3pr + 2q^2$: to this add $RA = qa^2 + pqa + ra + rp$; then will $PC - QB + RA = a^4 + 4ra + p^4 - 4p^2q + 4pr + 2q^2$: laſtly, ſubtract $4S = 4ra + 4s$, ſo ſhall $PC - QB + RA - 4S = a^4 + p^4 - 4p^2q + 4pr + 2q^2 - 4s = D$, the ſum of all the biquadrates.

In like manner (the laſt equation being, again, multiplied by $P = a + p$, the preceding one by $Q = pa + q$, &c. &c.) the ſum of the fifth powers will be found $= PD - QC + RB - SA + 5T$: from whence, and the preceding caſes, the law of continuation is manifeſt; the ſum (F) of the ſixth powers being $PE - QD + RC - SB + TA - 6U$; and the ſum (G) of the ſeventh powers $= PF - QE + RD - SC + TB - UA + 7W$, &c. &c.

But, if you would have the ſeveral values of B, C, D, E, &c. independent of one another, in terms of the given quantities P, Q, R, S, T, &c. then will

$B = P^2 - 2Q$,
$C = P^3 - 3PQ + 3R$,
$D = P^4 - 4P^2Q + 4PR + 2Q^2 - 4S$,

$E = P^5 - 5P^3Q + 5P^2R + 5PQ^2 - 5PS - 5QR + 5T$, *&c.* *&c.* which values may be continued on, at pleaſure, by multiplying the laſt by P, the laſt but one by − Q, the laſt but two by R, the laſt but three by − S, *&c.* and then adding all the products together; as is evident from the equations above derived.—These concluſions are of uſe in finding the limits of equations, and contain a demonſtration of a rule, given for that purpoſe, by Sir *Iſaac Newton*, in his *Univerſal Arithmetick*.

SECTION XII.

Of the Reſolution of Equations of ſeveral Dimenſions.

BEFORE we proceed to explain the methods of reſolving cubic, biquadratic, and other higher equations, it will be requiſite, in order to render that ſubject more clear and intelligible, to premiſe ſomething concerning the origin and compoſition of equations.

Mr. *Harriot* has ſhewn how equations are derived by the continued multiplication of binomial factors into each other: according to which method, ſuppoſing $x - a$, $x - b$, $x - c$, $x - d$, &c. to denote any number of ſuch factors, the value of x is to be ſo taken that ſome one of thoſe factors may be equal to nothing: then, if they be multiplied continually together, their product muſt alſo be equal to nothing, that is, $\overline{x-a} \times \overline{x-b} \times \overline{x-c} \times \overline{x-d}$ &c. $= 0$: in which equation x may, it is plain, be equal to any one of the quantities a, b, c, d, &c. ſince any one of theſe being ſubſtituted inſtead of x, the whole expreſſion vaniſhes. Hence it appears, that an equation may have as many roots as it has dimenſions, or as are expreſſed by the number of the factors, whereof it is ſuppoſed to be produced. Thus the quadratic equation $\overline{x-a} \times \overline{x-b} = 0$ or $x^2 \left.\begin{matrix}-a\\-b\end{matrix}\right\} x + ab = 0$, has two roots, a and b; the cubic equation $\overline{x-a} \times \overline{x-b} \times \overline{x-c} = 0$, or

$$x^3 + \left.\begin{matrix}-a\\-b\\-c\end{matrix}\right\} x^2 + \left.\begin{matrix}ab\\ac\\bc\end{matrix}\right\} x - abc = 0,$$

has three roots, a, b, and c; and the biquadratic equation, $\overline{x-a} \times \overline{x-b} \times \overline{x-c} \times \overline{x-d} = 0$, or

$$x^4 + \left.\begin{matrix}-a\\-b\\-c\\-d\end{matrix}\right\} x^3 + \left.\begin{matrix}ab+ac\\ad+bc\\bd+cd\end{matrix}\right\} x^2 + \left.\begin{matrix}-abc\\-abd\\-acd\\-bcd\end{matrix}\right\} x + abcd = 0,$$

has four roots, a, b, c, and d. From theſe equations it is obſervable, that the coefficient of the ſecond term is always equal to the ſum of all the roots, with contrary ſigns; that the coefficient of the third term is always equal to the ſum of their rectangles, or of all the products that can poſſibly ariſe by combining them, two and two; that the coefficient of the fourth is equal to the ſum of all their ſolids, or of all the products which can poſſibly ariſe, by combining them three and three; and that the laſt term of all, is produced by multiplying all the roots continually together. And all this, it is evident, muſt hold equally, when ſome of the roots are poſitive and the reſt negative, due regard being had to the ſigns. Thus, in the cubic equation

$$\overline{x-a} \times \overline{x-b} \times \overline{x+c} = 0, \text{ or } x^3 + \left.\begin{matrix}-a\\-b\\+c\end{matrix}\right\} x^2 + \left.\begin{matrix}+ab\\-ac\\-bc\end{matrix}\right\} x + abc = 0$$

(where two of the roots, a, b, are poſitive, and the other, $-c$, is negative) the coefficient of the ſecond term appears to be $-a-b+c$, and *that* of the third, $ab-ac-bc$, or $ab + a \times -c + b \times -c$, conformable to the preceding obſervations. Hence it follows, that, if one of the roots of an equation be given, the ſum of all the reſt will likewiſe be given; and that, in every equation where the ſecond term is wanting, the ſum of all the negative roots is exactly equal to *that* of all the poſitive ones; becauſe, in this caſe, they mutually deſtroy each other. But when the coefficient of the ſecond term is poſitive, then the negative roots, taken together, exceed the poſitive ones. But the negative roots, in any equation, may be changed to poſitive ones, and the poſitive to negative, by changing the ſigns of the ſecond, fourth, and ſixth terms, and ſo on alternately. Thus, the foregoing equation

$$(\overline{x-a}\times\overline{x-b}\times\overline{x+c}=)\,x^3+\left.\begin{matrix}-a\\-b\\+c\end{matrix}\right\}x^2\left.\begin{matrix}+ab\\-ac\\-bc\end{matrix}\right\}x+abc=0,$$

by changing the ſigns of the ſecond and fourth terms, becomes $x^3+\left.\begin{matrix}+a\\+b\\-c\end{matrix}\right\}x^2\left.\begin{matrix}+ab\\-ac\\-bc\end{matrix}\right\}x-abc=0$, or $\overline{x+a}\times\overline{x+b}\times\overline{x-c}=0$; where the roots, from $+a$, $+b$, and $-c$, are now become $-a$, $-b$, and $+c$. Moreover the negative roots may be changed to poſitive ones, or the poſitive to negative, by increaſing or diminiſhing each, by ſome known quantity. Thus in the quadratic equation $x^2+8x+15=0$, where the two roots are -3 and -5 (and therefore both negative) if $z-7$ be ſubſtituted for x, or which is the ſame, if each of the roots be increaſed by 7, the equation will become $\overline{z-7}|^2+8\times\overline{z-7}+15=0$; that is, $z^2-6z+8=0$, or $\overline{z-2}\times\overline{z-4}=0$; where the roots are 2 and 4, and therefore both poſitive. This method of augmenting, or diminiſhing the roots of an equation is ſometimes of uſe in preparing *it* for a ſolution, by taking away its ſecond term; which is always performed by adding, or ſubtracting $\frac{1}{2}$, $\frac{1}{3}$, or $\frac{1}{4}$ part, &c. of the coefficient of the ſaid term, according as the propoſed equation riſes to two, three, or four, &c. dimenſions. Thus, in the quadratic equation $x^2-8x+15=0$, let the roots be diminiſhed by 4, that is, let $x-4$ be put $=z$, or $x=4+z$; then, this value being ſubſtituted for x, the equation will become $\overline{z+4}|^2-8\times\overline{z+4}+15=0$, or $z^2-1=0$; in which the ſecond term is wanting.

Likewiſe, the cubic equation $x^3 - ax^2 + bx - c = 0$, by writing $x = -\frac{1}{3}a + z$, and proceeding as above, will become $x^3 * \left.\begin{matrix} +b \\ -\frac{1}{3}a^2 \end{matrix}\right\} x + \left.\begin{matrix} +\frac{1}{3}ab \\ -c \\ -\frac{2}{27}a^3 \end{matrix}\right\} = 0$; and ſo of others.

Hence it appears, how any affected quadratic may be reduced to a ſimple quadratic, and ſo reſolved without completing the ſquare; but this, by the bye. I now proceed to the matter propoſed, *viz.* the Reſolution of cubic, biquadratic, and other higher equations; and ſhall begin with ſhewing

How to determine whether ſome, or all the roots of an equation be rational, and, if ſo, what they are.

Find all the diviſors of the laſt term, and let them be ſubſtituted, one by one, for x, in the given equation; and then, if the poſitive and negative terms deſtroy each other, the diviſor ſo ſubſtituted is manifeſtly a root of the equation; but if none of the diviſors ſucceed, then the roots, for the general part, are either irrational or impoſſible: for the laſt term, as is ſhewn above, being always a multiple of all the roots, thoſe roots, when rational, muſt, neceſſarily, be in the number of its diviſors.

Examp. 1. Let the equation $x^3 - 4x^2 - 7x + 10 = 0$, be propoſed; then, the diviſors of (10) the laſt term being $+1, -1, +2, -2, +5, -5, +10, -10$, let theſe quantities be, ſucceſſively, ſubſtituted inſtead of x, and we ſhall have,

$1 - 4 - 7 + 10 = 0$, therefore 1 is a root;
$-1 - 4 + 7 + 10 = 12$, therefore -1 is no root;
$8 - 16 - 14 + 10 = -12$, therefore 2 is no root;
$-8 - 16 + 14 + 10 = 0$, therefore -2 is another root;
$125 - 100 - 35 + 10 = 0$, therefore 5 is the third root.

It ſometimes happens that the diviſors of the laſt term are very numerous; in which caſe, to avoid trouble, it will be convenient to transform the equation to another, wherein the diviſors are fewer; and this is beſt effected by increaſing or diminiſhing the roots by an unit, or ſome other known quantity.

Examp. 2. Let the equation propounded be $y^4 - 4y^3 - 8y + 32 = 0$; and, in order to change it to another whoſe laſt term admits of fewer diviſors, let $x + 1$ be ſubſtituted therein for y, and *it* will become $x^4 - 6x^2 - 16x + 21 = 0$; where the diviſors of the laſt term are, $1, -1, 3, -3, 7, -7, 21$, and -21; which being, ſucceſſively, ſubſtituted for x, as before, we have,

$1 - 6 - 16 + 21 = 0$, therefore 1 is one of the roots;
$1 - 6 + 16 + 21 = 32$, therefore -1 is not a root;
$81 - 54 - 48 + 21 = 0$, therefore 3 is another root.

But the other two roots, without proceeding further, will appear to be impoſſible; for, their ſum being equal to -4, the ſum of the two poſitive roots (already found), with a contrary ſign (as the ſecond term of the equation is here wanting), their product, therefore, cannot be equal to (7) the laſt term divided by the product of the other roots, as it would, if all the roots were poſſible. However, to get an expreſſion for theſe imaginary roots, let either of them be denoted by v, and the other will be denoted by $-4 - v$; which, multiplied together, give $-4v - v^2 = 7$; whence $v = -2 + \sqrt{-3}$, and conſequently $-4 - v = -2 - \sqrt{-3}$. Now let each of the four roots found above, be increaſed by unity, and you will have all the roots of the equation propoſed.

When the equation given is a *literal one*, you may ſtill proceed in the ſame manner, neglecting the known quantity and its powers, till you find what diviſors ſucceed; for each of *theſe*, multiplied by the ſaid quantity, will be a root of the equation. Thus, in the literal equation $x^3 + 3ax^2 - 4a^2x - 12a^3 = 0$, the numeral diviſors of the laſt term being $1, -1, 2, -2, 3, -3$, &c. I write theſe quantities, one by one, inſtead of x, not regarding a; and ſo have

$1 + 3 - 4 - 12 = -12$, therefore a is not a root;
$-1 + 3 + 4 - 12 = -6$, therefore $-a$ is no root;
$8 + 12 - 8 - 12 = 0$, therefore $2a$ is one of the roots;

$-8+12+8-12=0$, therefore $-2a$ is another root,
$27+27-12-12=30$, therefore $3a$ is not a root;
$-27+27+12-12=0$, therefore $-3a$ is the third root.

The reaſon of theſe operations is too obvious to need a further explanation. I ſhall here ſubjoin a different way, whereby the ſame concluſions may be derived, from Sir *Iſaac Newton*'s Method of Diviſors; which is thus:

Inſtead of the unknown quantity ſubſtitute, ſucceſſively, three, or more adjacent terms of the arithmetical progreſſion 2, 1, 0, — 1 — 2; *and, having collected all the terms of the equation into one ſum, let the quantities thus reſulting, together with all their diviſors, be placed in a line, right againſt the correſponding terms of the progreſſion* 2, 1, 0, — 1, — 2; *then ſeek among the diviſors an arithmetical progreſſion, whoſe terms correſpond with, or ſtand according to the order of the terms* 2, 1, 0, — 1, — 2, *of the firſt progreſſion, and whoſe common difference is either an unit, or ſome diviſor of the coefficient of the higheſt power of the unknown quantity* (x) *in the given equation. If any ſuch progreſſion can be diſcovered, let that term of it which ſtands againſt the term* 0, *in the firſt progreſſion, be divided by the common difference, and let the quotient, with the ſign* + *or* — *prefixed, according as the progreſſion is increaſing or decreaſing, be tried (as above) by ſubſtituting it for* x *in the propoſed equation.*

Thus, let the propoſed equation be $x^3 - x^2 - 10x + 6 = 0$; then, by ſubſtituting ſucceſſively the terms of the progreſſion 2, 1, 0, — 1, inſtead of x, there will ariſe — 10, — 4, 6, and 14, reſpectively; which, together with their diviſors, being placed right-againſt the correſponding terms of the progreſſion 2, 1, 0, — 1, the work will ſtand thus:

2	— 10	1 . 2 . 5 . 10	5
1	— 4	1 . 2 . 4 .	4
0	+ 6	1 . 2 . 3 . 6	3
— 1	+ 14	1 . 2 . 7 . 14	2

Now, ſince the coefficient of the higheſt power (x^3) is, here, only diviſible by an unit, I ſeek, among the diviſors, a collateral progreſſion whoſe common difference is an unit; and find the only one of this kind to be 5, 4, 3, 2; whoſe third term ſtanding againſt the term 0 in the firſt progreſſion, I therefore take and divide by unity, and then ſubſtitute the quotient, with a negative ſign, inſtead of x, and there reſults $-27-9+30+6=0$; therefore -3 is, manifeſtly, a root of the equation.

Again, if the propoſed equation were to be $2x^3-5x^2+4x-10=0$, we ſhall, by proceeding in the ſame manner, have

2	—6	1 . 2 . 3 . 6	1
1	—9	1 . 3 . 9	3
0	—10	1 . 2 . 5 . 10	5*
—1	—21	1 . 3 . 7 . 21	7
—2	—54	1 . 2 . 3 . 6 . 9 &c.	9

In which caſe, I diſcover, among the diviſors, the increaſing arithmetical progreſſion, 1, 3, 5, 7, 9; whoſe third term, 5, ſtanding againſt the term 0 in the firſt progreſſion, being divided by 2, the common difference, and the quotient $(\frac{5}{2})$ ſubſtituted for x, the buſineſs ſucceeds, the poſitive and negative terms deſtroying each other.

Moreover, if the equation $x^4+x^3-29x^2-9x+180=0$ were propoſed, the work will ſtand as follows:

2	70	1 . 2 . 5 . 7 . 10 . 14 . 35 . 70	1	2	5	7
1	144	1 . 2 . 3 . 4 . 6 . 8 . 9 . 12 &c.	2	3	4	6
0	180	1 . 2 . 3 . 4 . 5 . 6 . 9 . 10 &c.	3	4	3	5*
—1	160	1 . 2 . 4 . 5 . 8 . 10 . 16 . 20 &c.	4	5	2	4
—2	90	1 . 2 . 3 . 5 . 6 . 9 . 10 . 15 &c.	5	6	1	3

Here are diſcovered no leſs than four progreſſions, whoſe terms differ by unity; whereof the terms correſponding to the term 0, in the firſt progreſſion, are 3, 4, 3, and 5: therefore the two former progreſſions being aſcending ones, and the two latter deſcending, I try the quantities $+3$, $+4$, -3, -5, one by one, and find that they all ſucceed.

And after the ſame manner we may proceed in other caſes; but, in order to try whether any quantity thus found is a true root, we may, inſtead of ſubſtituting for x, divide the whole equation by that quantity connected to x, with a contrary ſign; for, if the diviſion terminates without a remainder, the ſaid quantity is manifeſtly a root of the equation.

Thus, in the laſt example, where the equation is $x^4 + x^3 - 29x^2 - 9x + 180 = 0$, the numbers to be tried being $+3$, $+4$, -3 and -5, I firſt take -3 and join it to x, and then divide the whole equation, $x^4 + x^3 - 29x^2 - 9x + 180$ $(= 0)$ by $x - 3$, the quantity thence ariſing, and find the quotient to come out $x^3 + 4x^2 - 17x - 60$, *exactly*. Therefore $+3$ is one of the roots.

Again, in order to try $+4$, the ſecond number, I divide the quotient, thus found, by $x - 4$, and there comes out $x^2 + 8x + 15$; therefore $+4$ is another root: laſtly, I try -3, by dividing the laſt quotient by $x + 3$, and find it alſo to ſucceed, the quotient being $x + 5$. See the operation at large.

$$\begin{array}{rllllll}
x-3) & x^4 & +\ x^3 & -29x^2 & -9x & +180 & (x^3+4x^2-17x-60 \\
 & x^4 & -3x^3 & & & & \\ \cline{2-3}
 & & +4x^3 & -29x^2 & & & \\
 & & +4x^3 & -12x^2 & & & \\ \cline{3-4}
 & & & -17x^2 & -\ 9x & & \\
 & & & -17x^2 & +51x & & \\ \cline{4-6}
 & & & & -60x & +180 & \\
 & & & & -60x & +180 & \\ \cline{5-6}
 & & & & 0 & 0 &
\end{array}$$

$$\begin{array}{rlllll}
x-4) & x^3 & +4x^2 & -17x & -60 & (x^2+8x+15 \\
 & x^3 & -4x^2 & & & \\ \cline{2-3}
 & & +8x^2 & -17x & & \\
 & & +8x^2 & -32x & & \\ \cline{3-4}
 & & & +15x & -60 & \\
 & & & +15x & -60 & \\ \cline{4-5}
 & & & 0 & 0 &
\end{array}$$

$x + 3)$

$$\begin{array}{l} x+3)\;x^2+8x+15\;(x+5 \\ \underline{x^2+3x} \\ +5x+15 \\ \underline{+5x+15} \\ 00 \end{array}$$

As another inſtance hereof, let there be propoſed the equation $2x^3 - 3x^2 + 16x - 24 = 0$; then expounding x by 2, 1, 0, and -1, ſucceſſively, and proceeding as in the foregoing examples, we have

2	+ 12	1 . 2 . 3 . 4 . 6 . 12	+ 2	− 1
1	− 9	1 . 3 . 9	+ 3	+ 1
0	− 24	1 . 2 . 3 . 4 . 6 . 8 &c.	+ 4	+ 3
− 1	− 45	1 . 3 . 5 . 9 . 15 . 45	+ 5	+ 5

Therefore, the quantities to be tried being 4 and $\frac{3}{2}$, I firſt attempt the diviſion by $x - 4$; which does not anſwer: but trying $x - \frac{3}{2}$, or (its double) $2x - 3$, I find it to ſucceed, the quotient being $x^2 + 8$, *exactly*.

The reaſon why the diviſors, thus found, do not always ſucceed, is, becauſe the firſt progreſſion 2, 1, 0, − 1 is not continued far enough, to know whether the correſponding progreſſion may not break off, after a certain number of terms; which it never can do when the buſineſs ſucceeds. Thus, in the laſt example, where we had two different progreſſions reſulting, had the operation, or ſeries, 2, 1, 0, − 1, been continued only two terms farther, you would have found the firſt of thoſe progreſſions to fail; whereas, on the contrary, the laſt (by which the buſineſs ſucceeds) will hold, carry on the progreſſion 2, 1, 0, − 1 as far as you will. The grounds of which, as well as of the whole method, upon which the foregoing obſervations are founded, may be explained in the following manner.

Let there be aſſumed any equation, as $ax^4 + bx^3 + cx^2 + dx + e = 0$, wherein a, b, c, d, and e, repreſent any whole numbers, poſitive or negative, and let $px + q$ denote any binomial diviſor by which the ſaid expreſſion

$ax^4 + bx^3 + cx^2 + dx + e$ is diviſible, and let the quotient thence ariſing be repreſented by $rx^3 + sx^2 + tx + v$, or, which is the ſame in effect, let $ax^4 + bx^3 + cx^2 + dx + e = \overline{px + q} \times \overline{rx^3 + sx^2 + tx + v}$. This being premiſed, ſuppoſe x to be now, ſucceſſively, expounded by the terms of the arithmetical progreſſion 2, 1, 0, — 1, — 2 *(as above)*; and then the correſponding values of our diviſor $px + q$, will, it is manifeſt, be expounded by $2p + q$, $p + q$, q, $-p + q$, and $-2p + q$ reſpectively; which alſo conſtitute an arithmetical progreſſion, whoſe common difference is p; which common difference *(p)* muſt be ſome diviſor of the coefficient *(a)* of the firſt term, otherwiſe the diviſion could not ſucceed, that is, p could not be had in a, without a remainder.

Hence it appears that the binomial diviſor, by which an expreſſion of ſeveral dimenſions is diviſible, muſt always vary as x varies, ſo as to be, ſucceſſively, expreſſed by the terms of an arithmetical progreſſion, whoſe common difference is ſome diviſor of the firſt, or higheſt term of that expreſſion.

It alſo appears, that the ſaid common difference is always the coefficient of the firſt term of the general diviſor; and that the term *(q)* of the progreſſion, which ariſes by taking $x = 0$, is the ſecond term. Therefore, whenever, by proceeding according to the method above preſcribed, a progreſſion is found, anſwering to the conditions here ſpecified, the terms of that progreſſion are to be conſidered only as ſo many ſucceſſive values of ſome general diviſor, as $px + q$. Whence the reaſon of the whole proceſs is manifeſt.

After the ſame manner we may proceed to the invention of trinomial diviſors, or diviſors of two dimenſions: for, let $mx^2 + px + q$, be any quantity of this kind, wherein m, p, and q repreſent whole numbers, poſitive or negative, and let the terms of the progreſſion 3, 2, 1, 0, — 1, — 2, — 3, be wrote therein, one by one, inſtead of x; whence it will become $9m + 3p + q$, $4m + 2p + q$, $m + p + q$, q, $m - p + q$, $4m - 2p + q$, and $9m - 3p + q$, reſpectively; where m muſt be ſome diviſor of the coefficient of the firſt term of the given expreſſion; otherwiſe, the diviſion could not ſucceed. Hence it appears,

1°, That the coefficient (m) of the firſt term of the diviſor muſt always be ſome numeral diviſor of the coefficient of the firſt term of the propoſed expreſſion.

2°, That the product of that coefficient by the ſquare of each of the terms of the aſſumed progreſſion, 3, 2, 1, 0, — 1, — 2, — 3, being ſubtracted from the correſponding value of the general diviſor, the remainders $(3p + q, 2p + q, p + q, q, -p + q, -2p + q, -3p + q)$ will be a ſeries of quantities in arithmetical progreſſion, whoſe common difference is the coefficient of the ſecond term of the diviſor.

3°, And that the term (q) of this progreſſion, which ariſes by taking $x = 0$, will always be the third, or laſt term of the ſaid diviſor. From whence we have the following rule. *Inſtead of x in the quantity propoſed, ſubſtitute, ſucceſſively, four or more adjacent terms of the progreſſion 3, 2, 1, 0, — 1, — 2, — 3; and from all the ſeveral diviſors of each of the numbers thus reſulting, ſubtract the ſquares of the correſponding terms of that progreſſion multiplied by ſome numeral diviſor of the higheſt term of the quantity propoſed, and ſet down the remainders right againſt the correſponding terms of the progreſſion 3, 2, 1, 0, — 1, — 2, — 3; and then ſeek out a collateral progreſſion which runs through theſe remainders; which being found, let a trinomial be aſſumed, whereof the coefficient of the firſt term is the foreſaid numeral diviſor; that of the ſecond term, the common difference of this collateral progreſſion; and whereof the third term is equal to that term of the ſaid progreſſion which ariſes by taking $x = 0$; and the expreſſion ſo aſſumed will be the diviſor to be tried. But it is to be obſerved that the ſecond term muſt have a negative or poſitive ſign, according as the progreſſion, found among the diviſors, is an increaſing or a decreaſing one.*

Thus, let the quantity propoſed be $x^4 - x^3 - 5x^2 + 12x - 6$; and then, by ſubſtituting 3, 2, 1, 0, — 1, — 2, ſucceſſively, inſtead of x, the numbers reſulting will be 39, 6, 1, — 6, — 21, and — 26 reſpectively; which, together with all their diviſors, both poſitive and negative, I place right-againſt the correſponding terms of the progreſſion 3, 2, 1, 0, — 1, — 2, in the following manner:

3	39	13	3	1	— 1	— 3	— 13	— 39
2	6	3	2	1	— 1	— 2	— 3	— 6
1	1	— 1						
0	6	3	2	1	— 1	— 2	— 3	— 6
— 1	21	7	3	1	— 1	— 3	— 7	— 21
— 2	26	13	2	1	— 1	— 2	— 13	— 26

Then, from each of theſe diviſors I ſubtract the ſquare of the correſponding term of the firſt progreſſion multiplied by unity (as being the only numeral diviſor of the firſt term), and the work ſtands thus:

3	30	4	— 6	— 8	— 10	— 12	— 22	— 48	+ 4	— 6
2	2	— 1	— 2	— 3	— 5	— 6	— 7	— 10	+ 2	— 3
1	0	— 2							+ 0	+ 0
0	6	3	2	1	— 1	— 2	— 3	— 6	— 2	+ 3*
— 1	20	6	2	0	— 2	— 4	— 8	— 22	— 4	+ 6
— 2	22	9	— 2	— 3	— 5	— 6	— 17	— 30	— 6	+ 9

Here I diſcover, among the remainders, two collateral progreſſions, *viz.* 4, 2, 0, — 2, — 4, — 6, and — 6, — 3, 0, + 3, + 6, + 9; therefore the quantity to be tried is either $x^2 + 2x - 2$, or $x^2 - 3x + 3$: by both of which the buſineſs ſucceeds.

This invention of trinomial diviſors is ſometimes of uſe in finding out the roots of an equation when they are irrational, or imaginary. Thus, let the equation given be $x^4 - 4x^3 + 5x^2 - 4x + 1 = 0$; and let x be ſucceſſively expounded by the terms of the progreſſion 3, 2, 1, 0, and the numbers reſulting will be 7, — 3, — 1 and 1; which, together with their diviſors, being ordered according to the preceding directions, the operation will ſtand as follows:

3	7	1	— 1	— 7	— 2	— 8	— 10	— 16	— 2	— 8
2	3	1	— 1	— 3	— 1	— 3	— 5	— 7	— 1	— 5
1	1	— 1	*	*	0	— 2	*	*	0	— 2
0	1	— 1	*	*	1	— 1	*	*	+ 1	+ 1*

Here we have two progreſſions, — 2, — 1, 0, 1; and — 8, — 5, — 2, 1; therefore the quantity to be tried is either $x^2 - x + 1$, or $x^2 - 3x + 1$; but I take the firſt, and having divided $x^4 - 4x^3 + 5x^2 - 4x + 1$ thereby, find *it* to ſucceed, the quotient coming out $x^2 - 3x + 1$, *exactly*. Therefore $x^4 - 4x^3 + 5x^2 - 4x + 1$ being univerſally equal to $\overline{x^2 - x + 1} \times \overline{x^2 - 3x + 1}$, let $x^2 - x + 1$ be taken $= 0$, and alſo $x^2 - 3x + 1 = 0$; from the former of which equations we have $x = \frac{1}{2} \pm \sqrt{-\frac{3}{4}}$; and from the latter $x = \frac{3}{2} \pm \sqrt{\frac{5}{4}}$. Therefore the four roots of the given equation are $\frac{1}{2} + \sqrt{-\frac{3}{4}}$, $\frac{1}{2} - \sqrt{-\frac{3}{4}}$, $\frac{3}{2} + \sqrt{\frac{5}{4}}$ and $\frac{3}{2} - \sqrt{\frac{5}{4}}$; whereof the two laſt are *irrational* and the two firſt *imaginary*. And in the ſame manner, the roots of a *literal equation*, as $z^4 - 4az^3 + 5a^2z^2 - 4a^3z + a^4 = 0$, where the terms are homogeneous, may be derived; for, let the roots be divided by a, that is, let x be put $= \frac{z}{a}$, or $ax = z$; and then, this value being ſubſtituted for z, the equation will become $x^4 - 4x^3 + 5x^2 - 4x + 1 = 0$; from which x will be found, as above; whence z $(= ax)$ is alſo known.

Having treated largely of the manner of managing ſuch equations as can be reſolved into rational factors, whether binomials, or trinomials, I come now to explain the more general methods, by which the roots of equations, of ſeveral dimenſions, are determined; and ſhall begin with

The Reſolution of cubic Equations, according to Cardan.

If the given equation has all its terms, the ſecond term muſt be taken away, as has been taught at the beginning of this ſection; and then the equation will be reduced to this form, *viz.* $x^3 + ax = b$; where a and b repreſent given quantities. Put $x = y + z$; and then, this value being ſubſtituted for x, our equation becomes $y^3 + 3y^2z + 3yz^2 + z^3 + a \times \overline{y + z} = b$, or $y^3 + z^3 + 3yz \times \overline{y + z} + a \times \overline{y + z} = b$. Aſſume, now, $3yz = -a$; ſo ſhall the terms $3yz \times \overline{y + z}$ and $a \times \overline{y + z}$ deſtroy each other, and our equation will be reduced to $y^3 + z^3 = b$. From the ſquare of which, let four times the cube of the equation $yz = -\frac{1}{3}a$ be ſubtracted, and we ſhall have $y^6 - 2y^3z^3 + z^6 = b^2 + \frac{4a^3}{27}$; and therefore, by extracting the ſquare root, on both ſides, $y^3 - z^3 = \sqrt{b^2 + \frac{4a^3}{27}}$; which added to, and ſubtracted from $y^3 + z^3 = b$, gives $2y^3 = b + \sqrt{b^2 + \frac{4a^3}{27}}$, and $2z^3 = b - \sqrt{b^2 + \frac{4a^3}{27}}$: hence $y = \overline{\frac{b}{2} + \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}}\Big|^{\frac{1}{3}}$, and $z = \overline{\frac{b}{2} - \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}}\Big|^{\frac{1}{3}}$; and conſequently x $(= y + z) = \overline{\frac{b}{2} + \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}}\Big|^{\frac{1}{3}} + \overline{\frac{b}{2} - \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}}\Big|^{\frac{1}{3}}$.

Which is *Cardan*'s Theorem: but the ſame thing may be exhibited in a manner rather more commodious for practice, by ſubſtituting for the ſecond term its equal $\frac{-\frac{1}{3}a}{\overline{\frac{b}{2} + \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}}\Big|^{\frac{1}{3}}}$ $\left(= \frac{-\frac{1}{3}a}{y} = z\text{, becauſe } yz = -\frac{1}{3}a\right)$. And this being done, our Theorem ſtands thus,

$$x = \overline{\frac{b}{2} + \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}}\Big|^{\frac{1}{3}} - \frac{\frac{1}{3}a}{\overline{\frac{b}{2} + \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}}\Big|^{\frac{1}{3}}}.$$

Example 1. Let the equation $y^3 + 3y^2 + 9y = 13$ be propounded; and, in order to deſtroy the ſecond term thereof, let $x - 1$ be put $= y$; ſo ſhall $\overline{x-1}|^3 + 3 \times \overline{x-1}|^2 + 9 \times \overline{x-1} = 13$, or $x^3 + 6x = 20$; therefore, in this caſe, a being $= 6$, and $b = 20$, we have x $\left(\overline{\frac{b}{2} + \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}}\Big|^{\frac{1}{3}} - \frac{\frac{1}{3}a}{\overline{\frac{b}{2} + \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}}\Big|^{\frac{1}{3}}}\right) =$

$\overline{10 + \sqrt{100 + 8}}|^{\frac{1}{3}} - \dfrac{2}{\overline{10 + \sqrt{100 + 8}}|^{\frac{1}{3}}} = \overline{20,3923}|^{\frac{1}{3}} - \dfrac{2}{\overline{20,3923}|^{\frac{1}{3}}} = 2.732 - .732 = 2$; and conſequently $y\ (= x - 1) = 1$.

Exam. 2. If the equation given be $y^6 - 3y^4 - 2y^2 - 8 = 0$; then, by writing $x + 1$ for y^2, it will become $\overline{x + 1}|^3 - 3 \times \overline{x + 1}|^2 - 2 \times \overline{x + 1} - 8 = 0$, or $x^3 - 5x = 12$: therefore, a being $= -5$, and $b = 12$, x will here be equal to $\overline{6 + \sqrt{36 - \frac{125}{27}}}|^{\frac{1}{3}} - \dfrac{-\frac{5}{3}}{\overline{6 + \sqrt{36 - \frac{125}{27}}}|^{\frac{1}{3}}} = \overline{6 + 5,6009}|^{\frac{1}{3}} + \dfrac{1,6666\ \&c.}{\overline{6 + 5,6009}|^{\frac{1}{3}}} = 2,26376 + ,73624 = 3$; and conſequently $y^2\ (= x + 1) = 4$: which is the only poſſible value of y^2 in the given equation. And it will be proper to take notice *here*, that this method is only of uſe in caſes where two, of the three roots, are impoſſible (except when they are equal); for $\dfrac{b^2}{4} + \dfrac{a^3}{27}$ being, in all other caſes, a negative quantity, it's ſquare root is manifeſtly impoſſible.

I ſhall now give the inveſtigation of the ſame general theorem, for the ſolution of cubics, by a different method; which is alſo applicable to other, higher equations.

Suppoſing, then, the ſum of two numbers, z and y, to be denoted by s, and their product (zy) by p, it will appear (*from Prob.* 68. *p.* 119) that the ſum of their cubes $(z^3 + y^3)$ will be truly expreſſed by $s^3 - 3ps$.

If, therefore, $z^3 + y^3$ be aſſumed $= b$, we ſhall alſo have $s^3 - 3ps = b$: but, zy being $= p$, or $y = \dfrac{p}{z}$, our firſt equation, $z^3 + y^3 = b$, will become $z^3 + \dfrac{p^3}{z^3} = b$; from which, by completing the ſquare, &c. z is found

$= \overline{\tfrac{1}{2}b+\sqrt{\tfrac{1}{4}bb-p^3}}\Big|^{\frac{1}{3}}$: whence $y\ (= \frac{p}{z})$ is given $=$ $\dfrac{p}{\overline{\tfrac{1}{2}b+\sqrt{\tfrac{1}{4}bb-p^3}}\Big|^{\frac{1}{3}}}$; and conſequently $s\ (= z+y)$ $= \overline{\tfrac{1}{2}b+\sqrt{\tfrac{1}{4}bb-p^3}}\Big|^{\frac{1}{3}} + \dfrac{p}{\overline{\tfrac{1}{2}b+\sqrt{\tfrac{1}{4}bb-p^3}}\Big|^{\frac{1}{3}}}$: which is, evidently, the true root of the equation $s^3 - 3ps = b$. From whence the root of the equation $x^3 + ax = b$, wherein the ſecond term is poſitive, will be given, by writing x for s, and $\frac{1}{3}a$ for $-p$; whence x is found

$$= \overline{\tfrac{1}{2}b+\sqrt{\frac{bb}{4}+\frac{a^3}{27}}}\Bigg|^{\frac{1}{3}} - \frac{\frac{1}{3}a}{\overline{\tfrac{1}{2}b+\sqrt{\frac{bb}{4}+\frac{a^3}{27}}}\Bigg|^{\frac{1}{3}}},$$

the ſame as before.

In like manner, if things be ſuppoſed as above, and there be, now, given $z^5 + y^5 = b$; then, *by the problem there referred to*, we likewiſe have $s^5 - 5ps^3 + 5p^2s = b$. But the firſt equation, by ſubſtituting $\frac{p}{z}$ for its equal y, becomes $z^5 + \frac{p^5}{z^5} = b$: whence $z^{10} - bz^5 = -p^5$, $z^5 = \frac{1}{2}b + \sqrt{\frac{1}{4}bb - p^5}$, and $z = \overline{\tfrac{1}{2}b+\sqrt{\tfrac{1}{4}bb-p^5}}\Big|^{\frac{1}{5}}$; and conſequently $s\ (= z + y = z + \frac{p}{z}) =$ $\overline{\tfrac{1}{2}b+\sqrt{\tfrac{1}{4}bb-p^5}}\Big|^{\frac{1}{5}} + \dfrac{p}{\overline{\tfrac{1}{2}b+\sqrt{\tfrac{1}{4}bb-p^5}}\Big|^{\frac{1}{5}}}$ = the true root of the equation $s^5 - 5ps^3 + 5p^2s = b$. Which by ſubſtituting x for s, and $-\frac{a}{5}$ for p, gives $x =$ $\overline{\tfrac{1}{2}b+\sqrt{\tfrac{1}{4}bb+\overline{\tfrac{1}{5}a}\big|^5}}\Big|^{\frac{1}{5}} - \dfrac{\frac{1}{5}a}{\overline{\tfrac{1}{2}b+\sqrt{\tfrac{1}{4}bb+\overline{\tfrac{1}{5}a}\big|^5}}\Big|^{\frac{1}{5}}}$, for the true root of the equation $x^5 + ax^3 + \frac{1}{5}a^2x = b$.

Generally, ſuppoſing $z^n + y^n = b$, or $z^n + \frac{p^n}{z^n} = b$ (becauſe $y = \frac{p}{z}$), we have $z^{2n} - bz^n = -p^n$; whence $z^n = \frac{1}{2}b + \sqrt{\frac{1}{4}bb - p^n}$, and $z = \overline{\frac{1}{2}b + \sqrt{\frac{1}{4}bb - p^n}}\Big|^{\frac{1}{n}}$: therefore $s\ (z+y) = z + \frac{p}{z} = \overline{\frac{1}{2}b + \sqrt{\frac{1}{4}bb - p^n}}\Big|^{\frac{1}{n}} + \frac{p}{\overline{\frac{1}{2}b + \sqrt{\frac{1}{4}bb - p^n}}\Big|^{\frac{1}{n}}}$; which is the true root of the equation $s^n - nps^{n-2} + n . \frac{n-3}{2} . p^2s^{n-4} - n . \frac{n-4}{2} . \frac{n-5}{3} . p^3s^{n-6} + n . \frac{n-5}{2} . \frac{n-6}{3} . \frac{n-7}{4} . p^4s^{n-8}$ — &c. $(= z^n + y^n) = b$.

This equation, by writing x for s, and $\frac{a}{n}$ for $-p$, becomes $x^n + ax^{n-2} + \frac{n-3}{2n} . a^2x^{n-4} + \frac{n-4}{2n} . \frac{n-5}{3n} . a^3x^{n-6} + \frac{n-5}{2n} . \frac{n-6}{3n} . \frac{n-7}{4n} . a^4x^{n-8}$ &c. $= b$; and its root $x = \overline{\frac{b}{2} + \sqrt{\frac{b^2}{4} + \frac{a^n}{n^n}}}\Big|^{\frac{1}{n}} - \frac{a}{n \times \overline{\frac{b}{2} + \sqrt{\frac{b^2}{4} + \frac{a^n}{n^n}}}\Big|^{\frac{1}{n}}}$. Wherein the two preceding Theorems are included, with innumerable others of the ſame kind; but as every one of them, except the firſt, requires a particular relation of the coefficients, ſeldom occurring in the Reſolution of problems, I ſhall take no further notice of them here, but proceed to

The Reſolution of biquadratic equations, according to Des Cartes.

Here the ſecond term is to be deſtroyed as in the ſolution of cubics; which being done, the given equation will be reduced to this form, $x^4 + ax^2 + bx + c = 0$; wherein a, b, and c may repreſent any quantities whatever, poſitive, or negative. Aſſume $\overline{x^2 + px + q} \times \overline{x^2 + rx + s} = x^4 + ax^2 + bx + c$; or, which is the ſame, let the biquadratic be conſidered, as produced by the multiplication of the two quadratics $x^2 + px + q = 0$, and $x^2 + rx + s = 0$: then, theſe laſt being actually multiplied into each other, we ſhall have $x^4 + ax^2 + bx$

$$+ c = x^4 \left.\begin{matrix} +p \\ +r \end{matrix}\right\} x^3 + \left.\begin{matrix} s \\ q \\ pr \end{matrix}\right\} x^2 + \left.\begin{matrix} ps \\ qr \end{matrix}\right\} x + qs;$$ whence,

by equating the homologous terms (in order to determine the value of the aſſumed coefficients, p, q, r, and s) we have $p + r = 0$, $s + q + pr = a$, $ps + qr = b$, and $qs = c$; from the firſt of which $r = -p$; from the ſecond $s + q\ (= a - pr) = a + p^2$; and from the third $s - q = \frac{b}{p}$. Now, by ſubtracting the ſquare of the laſt of theſe from that of the precedent, we have $4qs = a^2 + 2ap^2 + p^4 - \frac{bb}{pp}$, that is, $4c = a^2 + 2ap^2 + p^4 - \frac{bb}{pp}$ (becauſe $qs = c$); and therefore $p^6 + 2ap^4 \left.\begin{matrix} + a^2 \\ - 4c \end{matrix}\right\} p^2 = b^2$; from which p will be determined, as in example the ſecond, of the ſolution of cubics. Whence $s\ (= \frac{1}{2}a + \frac{1}{2}p^2 + \frac{b}{2p})$, and $q\ (= \frac{1}{2}a + \frac{1}{2}p^2 - \frac{b}{2p})$ are alſo known. And, by extracting the roots of the two aſſumed quadratics $x^2 + px + q = 0$, and

$x^2 + rx + s = 0$, we have x, in the one, $= -\frac{p}{2} \pm \sqrt{\frac{pp}{4} - q}$; and, in the other, $= -\frac{r}{2} \pm \sqrt{\frac{rr}{4} - s} = \frac{p}{2} \pm \sqrt{\frac{pp}{4} - s}$, becauſe $r = -p$. Therefore the four roots of the biquadratic, $x^4 + ax^2 + bx + c = 0$, are $\frac{p}{2} + \sqrt{\frac{pp}{4} - s}$, $\frac{p}{2} - \sqrt{\frac{pp}{4} - s}$, $-\frac{p}{2} + \sqrt{\frac{pp}{4} - q}$, and $-\frac{p}{2} - \sqrt{\frac{pp}{4} - q}$.

EXAMPLE.

Let the equation propounded be $y^4 - 4y^3 - 8y + 32 = 0$; then, to take away the ſecond term thereof, let $x + 1 = y$; whence, by ſubſtitution, $x^4 * - 6x^2 - 16x + 21 = 0$; which being compared with the general equation, $x^4 * + ax^2 + bx + c = 0$, we here have $a = -6$, $b = -16$, and $c = 21$; and conſequently $p^6 - 12p^4 - 48p^2 \left(= p^6 + 2ap^4 \left.\begin{matrix} +a^2 \\ -4c \end{matrix}\right\} p^2\right) = 256 \; (= b^2)$. Now, to deſtroy the ſecond term of this laſt equation alſo, make $z + 4 = p^2$; and then, this value being ſubſtituted, you will have $z^3 - 96z = 576$: whence, by the method above explained, z will be found $\left(= \overline{288 + \sqrt{\overline{288}|^2 - \overline{32}|^3}}\Big|^{\frac{1}{3}} + \frac{32}{\overline{288 + \sqrt{\overline{288}|^2 - \overline{32}|^3}}\Big|^{\frac{1}{3}}}\right) = 12$. Therefore $p \; (= \sqrt{z + 4})$ is $= 4$, $s \left(= \frac{a}{2} + \frac{p^2}{2} + \frac{b}{2p}\right) = 3$, and $q \left(= \frac{a}{2} + \frac{p^2}{2} - \frac{b}{2p}\right) = 7$; conſequently $\frac{p}{2} + \sqrt{\frac{pp}{4} - s} = 3$, $\frac{p}{2} - \sqrt{\frac{pp}{4} - s} = 1$, $-\frac{p}{2} +$

$\sqrt{\frac{pp}{4} - q} = -2 + \sqrt{-3}$, and $-\frac{p}{2} - \sqrt{\frac{pp}{4} - q}$ $= -2 - \sqrt{-3}$; which are the four roots of the equation $x^4 - 6x^2 - 16x + 21$; to each of which let unity be added, and you will have $4, 2, -1 + \sqrt{-3}$, and $-1 - \sqrt{-3}$, for the four roots of the equation propoſed; whereof the two laſt are impoſſible.

And that theſe roots are truly aſſigned, may be eaſily proved by multiplying the equations, $y - 4 = 0$, $y - 2 = 0$, $y + 1 - \sqrt{-3} = 0$, and $y + 1 + \sqrt{-3} = 0$, thus ariſing, continually together; for, from thence, the very equation given will be produced.

The reſolution of biquadratics by another method.

In the method of *Des Cartes*, above explained, all biquadratic equations are ſuppoſed to be generated from the multiplication of two quadratic ones: but, according to the way which I am now going to lay down, every ſuch equation is conceived to ariſe by taking the difference of two complete ſquares.

Here, the general equation $x^4 + ax^3 + bx^2 + cx + d = 0$ being propoſed, we are to aſſume $\overline{x^2 + \frac{1}{2}ax + A}^2 - \overline{Bx + C}^2 = x^4 + ax^3 + bx^2 + cx + d$; in which A, B, and C, repreſent unknown quantities, to be determined.

Then, $x^2 + \frac{1}{2}ax + A$, and $Bx + C$ being actually involved, we ſhall have

$$\left.\begin{array}{llllll} x^4 & + ax^3 & + 2Ax^2 & * & * \\ * & * & + \frac{1}{4}a^2x^2 & + aAx & + A^2 \\ * & * & - B^2x^2 & - 2BCx & - C^2 \end{array}\right\} = x^4 + ax^3 + bx^2$$

$+ cx + d$: from whence, by equating the homologous terms, will be given,

1. $2A + \frac{1}{4}a^2 - B^2 = b$, or, $2A + \frac{1}{4}a^2 - b = B^2$;
2. $aA - 2BC = c$, or, $aA - c = 2BC$;
3. $A - C^2 = d$, or, $A^2 - d = C^2$.

Let now the firſt and laſt of theſe equations be multiplied together, and the product will, evidently, be equal to $\frac{1}{4}$ of the ſquare of the ſecond, that is $2A^3 + \overline{\frac{1}{4}aa - b} \times A^2 - 2dA - d \times \overline{\frac{1}{4}aa - b}$ $(= B^2C^2) = \overline{\frac{1}{4} \times a^2A^2 - 2acA + c^2}$ $(= B^2C^2)$. Whence, denoting the given quantities $\frac{1}{4}ac - d$, and $\frac{1}{4}c^2 + d \times \overline{\frac{1}{4}aa - b}$ by k and l, reſpectively, there ariſes this cubic equation, $A^3 - \frac{1}{2}bA^2 + kA - \frac{1}{2}l = 0$: by means whereof the value of A may be determined (as hath been already taught); from which, and the preceding equations, both B and C will be known, B being given from thence $= \sqrt{2A + \frac{1}{4}aa - b}$, and $C = \frac{aA - c}{2B}$.

The ſeveral values of A, B, and C, being thus found, that of x will be readily obtained: for $\overline{x^2 + \frac{1}{2}ax + A}|^2 - \overline{Bx + C}|^2$ being univerſally, in all circumſtances of x, equal to $x^4 + ax^3 + bx^2 + cx + d$, it is evident, that when the value of x is taken ſuch, that the latter of theſe expreſſions becomes equal to nothing, the former muſt likewiſe be $= 0$; and conſequently $\overline{x^2 + \frac{1}{2}ax + A}|^2 = \overline{Bx + C}|^2$: whence, by extracting the ſquare root on both ſides, $x^2 + \frac{1}{2}ax + A = \pm Bx \pm C$; which, ſolved, gives $x = \pm \frac{1}{2}B - \frac{1}{4}a \pm \sqrt{\overline{\frac{1}{4}a \mp \frac{1}{2}B}|^2 \pm C - A}$ $= \pm \frac{1}{2}B - \frac{1}{4}a \pm \sqrt{\frac{1}{16}a^2 \mp \frac{1}{4}aB + \frac{1}{4}B^2 \pm C - A}$; exhibiting all the four different roots of the given equation, according to the variation of the ſigns.

This method will be found to have ſome advantages over that explained above. In the firſt place, there is no neceſſity *here* of being at the trouble of exterminating the ſecond term of the equation, in order to prepare it for a ſolution: ſecondly, the equation $A^3 - \frac{1}{2}bA^2 + kA - \frac{1}{2}l = 0$, here brought out, is of a more ſimple kind than that derived by the former method: and, thirdly (which advantage is the moſt conſiderable) the value of A, in this equation, will be *commenſurate* and *rational* (and therefore the eaſier to be diſcovered), not only when all the roots of the given equation are *commenſurate*, but when they are *irrational* and even *impoſſible*; as will appear from the examples ſubjoined.

Exam. 1. *Let there be given the equation* $x^4 + 12x - 17 = 0$.

Which being compared with the general equation $x^4 + ax^3 + bx^2 + cx + d = 0$, we have $a = 0$, $b = 0$, $c = 12$, and $d = -17$: therefore $k\ (\tfrac{1}{4}ac - d) = 17$, $l\ (\tfrac{1}{4}c^2 + d \times \overline{\tfrac{1}{4}aa - b}) = 36$; and conſequently $A^3 - \tfrac{1}{2}bA^2 + kA - \tfrac{1}{2}l = A^3 + 17A - 18 = 0$; where it is evident, by bare inſpection, that $A = 1$. Hence $B\ (= \sqrt{2A + \tfrac{1}{4}aa - b}) = \sqrt{2}$, $C\ (= \frac{aA - c}{2B}) = \frac{-12}{2\sqrt{2}} = -3\sqrt{2}$; and $x = \pm\tfrac{1}{2}\sqrt{2} \pm \sqrt{\tfrac{1}{2} \mp 3\sqrt{2} - 1} = \pm\tfrac{1}{2}\sqrt{2} \pm \sqrt{\mp 3\sqrt{2} - \tfrac{1}{2}}$. Therefore the four roots of the equation are $\tfrac{1}{2}\sqrt{2} + \sqrt{-3\sqrt{2} - \tfrac{1}{2}}$, $\tfrac{1}{2}\sqrt{2} - \sqrt{-3\sqrt{2} - \tfrac{1}{2}}$, $-\tfrac{1}{2}\sqrt{2} + \sqrt{3\sqrt{2} - \tfrac{1}{2}}$, and $-\tfrac{1}{2}\sqrt{2} - \sqrt{3\sqrt{2} - \tfrac{1}{2}}$; whereof the firſt and ſecond are impoſſible.

Exam. 2. *Let the equation given be* $x^4 - 6x^3 - 58x^2 - 114x - 11 = 0$.

Here $a = -6$, $b = -58$, $c = -114$, and $d = -11$; whence $k\ (\tfrac{1}{4}ac - d) = 182$, $l\ (\tfrac{1}{4}cc + d \times \overline{\tfrac{1}{4}aa - b}) = 2512$; and therefore $A^3 + 29A^2 + 182A - 1256 = 0$. Where, trying the diviſors 1, 2, 4, 157, &c. of the laſt term (according to the method delivered on p. 134) the third is found to ſucceed; the value of A being, therefore, $= 4$. Whence there is given $B = \sqrt{75} = 5\sqrt{3}$, $C = \frac{90}{10\sqrt{3}} = 3\sqrt{3}$, and $x\ (= \pm\tfrac{1}{2}B - \tfrac{1}{4}a \pm$

$\sqrt{\frac{3}{16}a^2 \mp \frac{1}{2}aB + \frac{1}{4}B^2 \pm C - A)} = \pm \frac{1}{2}\sqrt{3} + \frac{1}{2} \pm \sqrt{17 \pm \frac{21}{2}\sqrt{3}}$.

Exam. 3. *Let there be now propoſed the literal equation* $z^4 + 2az^3 - 37a^2z^2 - 38a^3z + a^4 = 0$.

This equation, by dividing the whole by a^4, and writing $x = \frac{z}{a}$, is reduced to the following numeral one, $x^4 + 2x^3 - 37x^2 - 38x + 1 = 0$. If, therefore, a, b, c, and d, be now expounded by 2, -37, -38, and 1, reſpectively, we ſhall here have k $(\frac{1}{2}ac - d) = -20$, l $(\frac{1}{4}c^2 + d \times \overline{\frac{1}{4}aa - b}) = 399$; and therefore, by ſubſtituting theſe values,

$$A^3 + \frac{37}{2}A^2 - 20A - \frac{399}{2} = 0,$$
$$\text{or, } 2A^3 + 37A^2 - 40A - 399 = 0.$$

Which equation, by the preceding methods, will be found to have three commenſurable roots, $\frac{7}{2}$, -3 and -19: and any one of theſe may be uſed, the reſult, take which you will, coming out exactly the ſame: Thus, by taking -3 for A, we ſhall have $x^2 + x - 3 = \pm\sqrt{2} \times \overline{4x + 2}$: but, if A be taken $= \frac{7}{2}$, then will $x^2 + x + \frac{7}{2} = \pm\sqrt{5} \times \overline{3x + \frac{1}{2}}$: laſtly, if A be taken $= -19$, then $x^2 + x - 19 = + 6\sqrt{10}$. All which are, in effect, but one and the ſame equation, as will readily appear by ſquaring both ſides of each, and properly tranſpoſing; whence the given equation $x^4 + 2x^3 - 37x^2 - 38x + 1 = 0$, will, in every caſe, emerge. And the ſame obſervation extends to all other caſes, where there are more roots than one; it being indifferent which value we uſe; unleſs, that ſome are to be preferred, as being the moſt ſimple and commodious.

Having given the general ſolution of biquadratic equations, by the means of cubic ones, I ſhall now point out two or three particular caſes, where every thing may be performed by the reſolution of a quadratic only.

Theſe are diſcovered from the preceding equations,

$$2A + \tfrac{1}{4}a^{2} - b = B^{2},$$
$$aA - c = 2BC$$
$$\text{and } A^{2} - d = C^{2}:$$

wherein, *if* A *be ſuppoſed* = 0, it is plain that $\tfrac{1}{4}a^{2} - b = B^{2}$, $-c = 2bC$, and $-d = C^{2}$: whence $B = \sqrt{\tfrac{1}{4}aa - b}$, $C = \dfrac{-c}{2\sqrt{\tfrac{1}{4}aa - b}} = \sqrt{-d}$, and conſequently $d = \dfrac{cc}{4f}$; by making $f = b - \tfrac{1}{4}aa$.

Therefore, in this caſe, (wherein $d = \dfrac{cc}{4f}$) the general equation $x^{2} + \tfrac{1}{2}ax + A = \pm Bx \pm C$, will become $x^{2} + \tfrac{1}{2}ax = \pm x\sqrt{-f} \mp \sqrt{-d}$.

But, *if* B *be ſuppoſed* = 0; then will $2A + \tfrac{1}{4}a^{2} - b = 0$, and alſo $aA - c = 0$; whence $A = \tfrac{1}{2}b - \tfrac{1}{8}a^{2} = \tfrac{1}{2}f = \dfrac{c}{a}$; and therefore $C \left(= \sqrt{A^{2} - d}\right) = \sqrt{\tfrac{1}{4}ff - d}$: ſo that in this caſe (where $c = \dfrac{af}{2}$) the general equation becomes $x^{2} + \tfrac{1}{2}ax + \tfrac{1}{2}f = \pm\sqrt{\tfrac{1}{4}ff - d}$; which, ſolved, gives $x = -\tfrac{1}{4}a \pm \sqrt{\left.\tfrac{1}{4}a\right|^{2} - \tfrac{1}{2}f \pm \sqrt{\tfrac{1}{4}ff - d}}$.

Laſtly, *if* C *be ſuppoſed* = 0, then will $aA - c = 0$, and $A^{2} - d = 0$; conſequently $A = \dfrac{c}{a} = \sqrt{d}$, and $B \left(= \sqrt{2A + \tfrac{1}{4}a^{2} - b}\right) = \sqrt{\dfrac{2c}{a} - f}$: therefore, in this caſe (where $d = \dfrac{cc}{aa}$) we ſhall have $x^{2} + \tfrac{1}{2}ax + \dfrac{c}{a} = \pm x\sqrt{\dfrac{2c}{a} - f}$.

From the whole of which it appears, that, if c be $= \dfrac{af}{2}$, or d, either, equal to $\dfrac{cc}{4f}$, or to $\dfrac{cc}{aa}$ (f being =

$b-$

$b - \frac{3}{8}aa)$; then the roots of the given equation, $x^{4} + ax^{3} + bx^{2} + cx + d = 0$, may be obtained, by the reſolution of a quadratic, only.

Exam. 1. *Let there be given* $x^{4} * - 25x^{2} + 60x - 36 = 0$.

Here $a = 0$, $b = -25$, $c = 60$, and $d = -36$; therefore, $f\ (= -25)$ being $= \frac{cc}{4d}\ (= -25)$, we have, *by caſe* 1, $x^{2} + \frac{1}{2}ax = \pm x\sqrt{-f} \mp \sqrt{-d}$; that is, $x^{2} = \pm 5x \mp 6$: which, ſolved, gives $x = \pm \frac{5}{2} \pm \sqrt{\frac{25}{4} \mp 6}$, that is, $x = \frac{5}{2} \pm \frac{1}{2}$, or, $x = -\frac{5}{2} \pm \frac{7}{2}$: ſo that 3, 2, 1, and -6, are the four roots of the equation propounded.

Exam. 2. *Let there be now given* $x^{4} + 2qx^{3} + 3q^{2}x^{2} + 2q^{3}x - r^{4} = 0$.

Then, a being $= 2q$, $b = 3q^{2}$, $c = 2q^{3}$, and $d = -r^{4}$, thence will $f\ (= b - \frac{1}{4}aa) = 2q^{2}$, and $\frac{af}{2}\ (= 2q^{3}) = c$; and ſo, the example belonging to *caſe* 2, we have $x\ (= -\frac{1}{4}a + \sqrt{\overline{\frac{1}{4}a}^{2} - \frac{1}{2}f \pm \sqrt{\frac{1}{4}ff - d}}) = -\frac{1}{2}q \pm \sqrt{-\frac{3}{4}qq \pm \sqrt{q^{4} + r^{4}}}$.

Exam. 3. *Laſtly, ſuppoſe there to be given the equation* $x^{4} - 9x^{3} + 15x^{2} - 27x + 9 = 0$.

Here, a being $= -9$, $b = 15$, $c = -27$, and $d = 9$, it is evident that $\frac{cc}{a}\ (= 9) = d\ (= 9)$: therefore, *by caſe* 3, we have $x^{2} + \frac{1}{2}ax + \frac{c}{a} = \pm x\sqrt{\frac{2c}{a} + \frac{1}{4}aa - b}$;

that is, $x^2 - 4\frac{1}{2}x + 3$ $(= \pm x\sqrt{6 + \frac{81}{4} - 15}) = \pm \frac{3}{2}x\sqrt{5}$: which, ſolved, gives

$$x = \frac{9 \pm 3\sqrt{5} \pm \sqrt{78 \pm 54\sqrt{5}}}{4}.$$

The Reſolution of LITERAL EQUATIONS, *wherein the given, and the unknown quantity, are alike affected.*

Equations of this kind, in which the given and the unknown quantities can be ſubſtituted, alternately, for each other, without producing a new equation, are always capable of being reduced to others of lower dimenſions. *In order to ſuch a reduction let the equation, if it be of an even dimenſion, be firſt divided by the equal powers of its two quantities in the middle term: then aſſume a new equation, by putting ſome quantity (or letter) equal to the ſum of the two quotients that ariſe by dividing thoſe quantities one by the other, alternately; by means of which equation, let the ſaid quantities be exterminated; whence a numeral equation will emerge, of half the dimenſions with the given literal one.*

But if the equation propounded be of an odd dimenſion, let it *be,* firſt, *divided by the ſum of its two quantities, ſo will* it *become of an even dimenſion, and its reſolution will therefore depend upon the preceding rule.*

Exam. 1. *Let there be given the equation* $x^4 - 4ax^3 + 5a^2x^2 - 4a^3x + a^4 = 0$.

Here, dividing by a^2x^2, we have $\frac{xx}{aa} + \frac{4x}{a} + 5 - \frac{4a}{x} + \frac{aa}{xx} = 0$ (or $\frac{xx}{aa} + \frac{aa}{xx} - 4 \times \overline{\frac{x}{a} + \frac{a}{x}} + 5 = 0$, by joining the correſponding terms); and by making $z = \frac{x}{a} + \frac{a}{x}$, and ſquaring both ſides, we have alſo $z^2 = \frac{xx}{aa} + 2 + \frac{aa}{xx}$, or $z^2 - 2 = \frac{xx}{aa} + \frac{aa}{xx}$. Therefore, by ſubſtituting theſe values, our equation becomes $z^2 - 2 - 4z + 5 = 0$, or $z^2 - 4z = -3$; whence $z = 3$. But $\frac{x}{a} + \frac{a}{x}$ being $= z$, we have $x^2 - zax = -a^2$; and conſequently $x = \frac{1}{2}za \pm \sqrt{\frac{1}{4}a^2z^2 - aa} = \frac{1}{2}a \times \overline{z \pm \sqrt{zz-4}} = \frac{1}{2}a \times \overline{3 \pm \sqrt{5}}$, *in the preſent caſe.*

Exam. 2. *Let there be given* $x^5 + 4ax^4 - 12a^2x^3 - 12a^3x^2 + 4a^4x + a^5 = 0$.

In this caſe we muſt firſt divide by $x + a$, and the quotient will come out $x^4 + 3ax^3 - 15a^2x^2 + 3a^3x + a^4 = 0$: whence, by proceeding as in the former example, we have $\frac{xx}{aa} + \frac{aa}{xx} + 3 \times \overline{\frac{x}{a} + \frac{a}{x}} - 15 = 0$, or $z^2 - 2 + 3z - 15 = 0$, and from thence $z = \frac{\sqrt{77} - 3}{2}$.

Exam. 3. *Suppoſe there to be given* $7x^6 - 26ax^5 - 26a^5x + 7a^6 = 0$.

Which, divided by a^3x^3, becomes $7 \times \overline{\frac{x^3}{a^3} + \frac{a^3}{x^3}} - 26 \times \overline{\frac{x^2}{a^2} + \frac{a^2}{x^2}} = 0$. Now, making, as before, $z = \frac{x}{a} + \frac{a}{x}$, we have $z^2 - 2 = \frac{x^2}{a^2} + \frac{a^2}{x^2}$; and, multiplying again by $z = \frac{x}{a} + \frac{a}{x}$, we likewiſe have $z^3 - 2z = \frac{x^3}{a^3} + \frac{a}{x} + \frac{x}{a} + \frac{a^3}{x^3} = \frac{x^3}{a^3} + z + \frac{a^3}{x^3}$; and therefore, $z^3 - 3z = \frac{x^3}{a^3} + \frac{a^3}{x^3}$: which values being ſubſtituted above, our equation becomes $7 \times \overline{z^3 - 3z} - 26 \times \overline{z^2 - 2} = 0$, or, $7z^3 - 26z^2 - 21z + 52 = 0$.

Where, trying the diviſors of the laſt term, which are 1, 2, 4, 13, &c. the third is found to anſwer; z, conſequently, being $= 4$.

Exam. 4. *Wherein let there be given* $2x^7 - 13a^2x^5 - 13a^5x^2 + 2a^7 = 0$.

Here dividing, firſt, by $x + a$, the quotient will be $2x^6 - 2ax^5 - 11a^2x^4 + 11a^3x^3 - 11a^4x^2 - 2a^5x + 2a^6 = 0$; which, divided again by a^3x^3, gives $2 \times \overline{\frac{x^3}{a^3} + \frac{a^3}{x^3}} - 2 \times \overline{\frac{x^2}{a^2} + \frac{a^2}{x^2}} - 11 \times \overline{\frac{x}{a} + \frac{a}{x}} + 11 = 0$; that is, $2 \times \overline{z^3 - 3z} - 2 \times \overline{z^2 - 2} - 11z + 11 = 0$, or $2z^3 - 2z^2 - 17z + 15 = 0$ (*vid. p.* 119): whence $z = 3$.

A *literal equation* may be made to correſpond with a *numeral one*, by ſubſtituting an unit in the room of the given quantity *(or letter):* and equations that do not ſeem, at firſt, to belong to the preceding *claſs*, may ſometimes be reduced to ſuch, by a proper ſubſtitution; that is, by putting the quotient of the firſt term divided by the laſt, equal to ſome new unknown quantity (or letter) raiſed to the power expreſſing the dimenſion of the equation. Thus, if the equation given be $2x^4 + 24x^3 - 315x^2 + 216x + 162 = 0$; by putting $\frac{2x^4}{162} = y^4$, we have $x = 3y$; whence, after ſubſtitution, the given equation becomes $162y^4 + 648y^3 - 2835y^2 + 648y + 162 = 0$: which now anſwers to the rule, and may be reduced down to $2y^4 + 8y^3 - 35y^2 + 8y + 2 = 0$.

Of the Reſolution of Equations by approximation and converging ſeries's.

The methods hitherto given, for finding the roots of equations, are, either, very troubleſome and laborious, or elſe confined to particular caſes; but *that* by converging ſeries's, which we are here going to explain, is univerſal, extending to all kinds of equations; and, though not accurately true, gives the value ſought, with little trouble, to a very great degree of exactneſs. When an equation is propoſed to be ſolved by this method, the root thereof muſt, firſt of all, be nearly eſtimated (which, from the nature of the problem and a few trials, may, in moſt caſes, be very eaſily done); and ſome letter, or unknown quantity (as z) muſt be aſſumed, to expreſs the difference between that value, which we will call r, and the true value (x); then, inſtead of x, in the given equation, you are to ſubſtitute its equal $r \pm z$, and there will emerge a new equation, affected only with z and known quantities; wherein all the terms having two, or more dimenſions of z, may be rejected, as inconſiderable in reſpect of the reſt; which being done, the value of z will be found, by the reſolution of a ſimple equation; from whence that of x ($= r \pm z$) will alſo be known. But, if this value ſhould not be thought ſufficiently near the truth, the operation may be repeated, by ſubſtituting the ſaid value inſtead of r, in the equation exhibiting the value of z; which will give a ſecond correction for the value of x.

As an example hereof, let the equation $x^3 + 10x^2 + 50x = 2600$, be propoſed: then, ſince it appears that x muſt, in this caſe, be ſomewhat greater than 10, let r be put $= 10$, and $r + z = x$; which value being ſubſtituted for x, in the given equation, we have $r^3 + 3r^2z + 3rz^2 + z^3 + 10r^2 + 20rz + 10z^2 + 50r + 50z = 2600$: this, by rejecting all the terms wherein two or more dimenſions of z are concerned, is reduced to $r^3 + 3r^2z + 10r^2 + 20rz + 50r + 50z = 2600$; whence z comes out $= \frac{2600 - r^3 - 10r^2 - 50r}{3r^2 + 20r + 50} = 0.18$, nearly: which, added to 10 ($= r$), gives 10,18 for the value of x. But, in order to repeat the operation, let this value be ſubſtituted for r, in the laſt equation, and you will have $z = -,0005347$; which, added to 10,18, gives 10,1794653, for the value of x, a ſecond time corrected. And, if this laſt value be, again, ſubſtituted for r, you will have a third correction of x; from whence a fourth may, in like manner, be found; and ſo on, until you arrive to what degree of exactneſs you pleaſe.

But, in order to get the general equation from whence theſe ſucceſſive corrections are derived, with as little trouble as poſſible, you may neglect all thoſe terms, which, in ſubſtituting for x and its powers, would riſe to two or more dimenſions of the converging quantity: for, they being, by the rule, to be omitted, it is better entirely to exclude them, than to take them in, and afterwards reject them.

Thus, in the equation $x^3 + x^2 + x = 90$, let $r + z$ be put $= x$; and then, by omitting all the powers of z above the firſt, we ſhall have $r^2 + 2rz = x^2$, and $r^3 + 3r^2z = x^3$, nearly; which, ſubſtituted above, give $r^3 + 3r^2z + r^2 + 2rz + r + z = 90$; whence z is found $= \frac{90 - r^3 - r^2 - r}{3r^2 + 2r + 1}$. Therefore, if r be now taken equal to 4 (which, it is eaſy to perceive, is nearly the true value of x) we ſhall have z $(= \frac{90 - 64 - 16 - 4}{48 + 8 + 1} = \frac{6}{57}) =$ 0.10 &c. which, added to 4, gives 4.1, for the value of x, once corrected: and, if this value of x be now ſubſtituted for r, we ſhall have z $(= \frac{90 - r^3 - r^2 - r}{3r^2 + 2r + 1}) = ,00283$; which, added to 4.1, gives 4.10283, for the value of x, a ſecond time corrected.

In the ſame manner, a general theorem may be derived, for equations of any number of dimenſions. Let $ax^n + bx^{n-1} + cx^{n-2} + dx^{n-3} + ex^{n-4}$ &c. $=$ Q, be ſuch an equation, where n, a, b, c, d &c. repreſent any given quantities, poſitive, or negative: then, putting $r + z = x$, we have, by the Theorem in p. 41.

$$\begin{aligned} x^n &= r^n + nr^{n-1}z \text{ \&c.} \\ x^{n-1} &= r^{n-1} + \overline{n-1} \times r^{n-2}z \text{ \&c.} \\ x^{n-2} &= r^{n-2} + \overline{n-2} \times r^{n-3}z \text{ \&c.} \\ &\text{\&c.} \end{aligned}$$

Which values being ſubſtituted in the propoſed equation, *it* becomes $ar^n + nar^{n-1}z + br^{n-1} + \overline{n-1} \times br^{n-2}z + cr^{n-2} + \overline{n-2} \times cr^{n-3}z + dr^{n-3} + \overline{n-3} \times dr^{n-4}z$ &c. $= Q$. From which z is found $=$

$$\frac{Q - ar^n - br^{n-1} - cr^{n-2} - dr^{n-3} - er^{n-4} \text{ \&c.}}{nar^{n-1} + \overline{n-1} \times br^{n-2} + \overline{n-2} \times cr^{n-3} + \overline{n-3} \times dr^{n-4} + \overline{n-4} \times er^{n-5} \text{ \&c.}}$$

As an inſtance of the uſe of this Theorem, let the equation $-x^3 + 300x = 1000$ be propounded. Here n being $= 3$, $a = -1$, $b = 0$, $c = 300$, and $Q = 1000$, we ſhall, by ſubſtituting theſe values above, have $z = \frac{1000 + r^3 - 300r}{-3r^2 + 300}$: in which (as it appears, by inſpection, that one of the values of x muſt be greater than 3, but leſs than 4) let r be taken $= 3$; and z will become $= \frac{127}{273} = 0.5$, and conſequently x $(= r + z) = 3.5$, *nearly*. Therefore, to repeat the operation, let 3.5 be now wrote inſtead of r, and z will come out $= \frac{-7,125}{263.25} = -0,027$; which, added to 3.5, gives 3.473, for the value of x, *twice corrected*. And, by repeating the operation once more, x will be found $= 3,47296351$; which is true to the laſt figure.

If the root of a pure power be to be extracted, or, which is the ſame, if the propoſed equation be $x^n = Q$; then, a being $= 1$, and b, c, d, &c. each $= 0$; z, in this caſe, will be barely $= \frac{Q - r^n}{nr^{n-1}}$; which may ſerve as a general Theorem for extracting the roots of pure powers. Thus, if it were required to extract the cube root of 10; then, n being $= 3$, and $Q = 10$, z will be $= \frac{10 - r^3}{3r^2}$; in which, let r be taken $= 2$, and *it* will become $z = \frac{2}{12} = 0,16$: therefore $x = 2,16$; from

whence,

whence, by repeating the operation, the next value of x will be found $= 2{,}1544$.

The manner of approximating hitherto explained, as all the powers of the converging quantity after the firſt are rejected, only doubles the number of figures at every operation. But I ſhall now give the inveſtigation of other rules, or *formulæ*, whereby the number of places may be tripled, quadrupled, or even quintupled, at every operation.

Let there be aſſumed the general equation $az + bz^2 + cz^3 + dz^4$ &c. $= p$; z, as above, being the converging quantity, and a, b, c, d, &c. ſuch known numbers as ariſe by ſubſtituting in the original equation, after the value of the required root is nearly eſtimated.

Then, by tranſpoſition and diviſion, we ſhall have $z = \frac{p}{a} - \frac{bz^2}{a} - \frac{cz^3}{a} - \frac{dz^4}{a}$ &c. from whence, by rejecting all the terms after the firſt, and writing $q = \frac{p}{a}$ there will be given $z = q$: which value, taking in only one term of the given ſeries, I call an approximation of the firſt degree, or order.

To obtain an approximation of the ſecond degree, or ſuch a one as ſhall include two terms of the ſeries, let the value of z found above, be now ſubſtituted in the ſecond term $\frac{bz^2}{a}$, rejecting all the following ones; ſo ſhall $z = \frac{p}{a} - \frac{bq^2}{a} = q - \frac{bq^2}{a}$; which triples the number of figures at every operation.

For an approximation of the third degree, let this laſt value of z be now ſubſtituted in the ſecond and third terms, neglecting every where all ſuch quantities as have more than three dimenſions of q: whence z will be had $\left(= q - \frac{bq^2}{a} + \frac{2b^2q^3}{aa} - \frac{cq^3}{a}\right) = q - \frac{b}{a}q^2 + \frac{2bb - ac}{aa}q^3$.

The manner of continuing theſe approximations is ſufficiently evident: but there are others, of the ſame degrees, differing in form, which are rather more commodious; and whereof the inveſtigation is alſo ſomewhat different.

It is evident from the given equation, that $z = \frac{p}{a + bz + cz^2 + dz^3 \&c.}$. If, therefore, the firſt value of z, found above, be ſubſtituted in the denominator, and all the terms after the ſecond be rejected, we ſhall have $z = \frac{p}{a + bq} = \frac{ap}{aa + bp}$; which is an approximation of the ſecond degree.

But, if, for z you write its ſecond value, $q - \frac{bq^2}{a}$, you will then have z $\left(= \frac{p}{a + bq - \frac{b^2q^2}{a} + cq^2}\right) = \frac{p}{a + bq - \overline{\frac{bb}{a} - c} \cdot q^2}$; being an approximation of the third degree.

Again, by writing $q - \frac{b}{a} q^2 + \frac{2bb - ac}{aa} \cdot q^3$ in the room of z, and neglecting every where all ſuch terms as have more than 3 dimenſions of q, you will have

$$z \left(= \frac{p}{a + bq - \frac{bb}{a} q^2 + \frac{2b^3 - abc}{aa} \cdot q^3 + c \times \overline{q^2 - \frac{2b}{a} \cdot q^3} + dq^3}\right)$$

$$= \frac{p}{a + bq - \overline{\frac{bb}{a} - c} \cdot q^2 + \overline{\frac{2b^3}{aa} - \frac{3bc}{a} + d} \cdot q^3}:$$

which is an approximation of the fourth degree.

It is obſervable, that the powers of the converging quantity q, in the former approximations, ſtand, all of them, in the numerator; but *here*, in the denominator:

but there is an artifice for bringing them, alike, into both, and thereby leſſening the number of dimenſions, without taking away from the rate of convergency.

To begin with the approximation $z =$

$$\frac{p}{a + bq - \overline{\frac{bb}{a} - c} \cdot q^2}$$, which is of the third degree,

put $s = \frac{b}{a} - \frac{c}{b} =$ the coefficient of the laſt term of the denominator divided by that of the laſt but one; ſo ſhall $z = \frac{p}{a + bq - bsq^2}$; whereof the numerator and the denominator being, equally, multiplied by $\overline{1 + sq}$, it becomes $z = \frac{p \times \overline{1 + sq}}{a + bq - bsq^2 + asq + bsq^2 - bs^2q^3}$: but, the approximation being only of the third degree, bs^2q^3 may be rejected, and ſo we have

$$z = \frac{p + pqs}{a + \overline{b + as} \cdot q} = \frac{\overline{a + sp} \cdot p}{aa + \overline{b + as} \cdot p}.$$

In the ſame manner, in order to exterminate the third dimenſion of q out of the equation

$$z = \frac{p}{a + bq - \overline{\frac{bb}{a} - c} \cdot q^2 + \overline{\frac{2b^3}{aa} - \frac{3bc}{a} + d} \cdot q^3},$$

put $w = \frac{2b}{a} + \frac{ad - bc}{bb - ac} =$ the coefficient of the laſt term of the denominator divided by that of the laſt but one; then will $z = \frac{p}{a + bq - \overline{\frac{bb}{a} - c} \cdot q^2 + \overline{\frac{bb}{a} - c} \cdot wq^3}$

$= \frac{p}{a + bq - bsq^2 + bswq^3}$ (becauſe $s = \frac{b}{a} - \frac{c}{b}$); whereof the terms being equally multiplied by $\overline{1 + wq}$, &c. we thence have $z = \frac{p \times \overline{1 + wq}}{a + bq - bsq^2 + awq + bwq^2}$

$=$

$$= \frac{p \times \overline{1 + wq.}}{a + \overline{b + aw} \,.\, q + \overline{w - s} \,.\, bq^2} =$$

$\frac{ap \times \overline{a + wp}}{a \times \overline{aa + \overline{b + aw} \,.\, p + \overline{w - s} \,.\, pp}}$: which is an approximation of the fourth degree, and quintuples the number of figures at every operation.

By purſuing the ſame method, other equations might be determined to include 5, or more terms of the given ſeries; but, then, they would be found more tedious, and perplexed in proportion; ſo that no real advantage, in practice, could be reaped therefrom. I ſhall, therefore, proceed now to illuſtrate *what* is laid down above by a few examples.

Exam. 1. *Let the equation given be* $x^2 + 20x = 100$.

Here, x appearing, by inſpection, to be ſomething greater than 4, make $4 + z = x$; then the given equation, by ſubſtitution, becomes $28z + z^2 = 4$. Therefore, in this caſe, $a = 28$, $b = 1$, $c = 0$, &c. and $p = 4$; and conſequently $\frac{ap}{aa + bp}$ $(= \frac{112}{788} = \frac{28}{197}) =$ 0.14213; which is one approximation of the value of z. But, if greater exactneſs be deſired, then s $(\frac{b}{a} - \frac{c}{b})$ being here $\frac{1}{28}$, and w $(\frac{2b}{a} + \frac{ad - bc}{bb - ac}) = \frac{1}{14}$, we ſhall, according to our two laſt *formulæ*, have

$$z \left(= \frac{\overline{a + sp} \,.\, p}{aa + \overline{b + as} \,.\, p}\right) = \frac{\overline{28 + \frac{1}{7}} \times 4}{28 \times 28 + 2 \times 4} =$$

$\frac{28 + \frac{1}{7}}{28 \times 7 + 2} = \frac{197}{1386} =$ 0,14213564, nearly; and

$$z\left(= \frac{ap \times \overline{a + wp}}{a \times \overline{aa + \overline{b + aw} \,.\, p + \overline{w - s} \,.\, pp}}\right) = \frac{28 \times 4 \times \overline{28 + \frac{2}{7}}}{28 \times \overline{784 + 12 + \frac{4}{7}}}$$

$$= \frac{28 \times \overline{28 + \frac{2}{7}}}{7 \times 796 + \frac{1}{7}} = \frac{28 \times 198}{49 \times 796 + 1} = \frac{5544}{39005} =$$

0.1421356236, more nearly; which value is true to the laſt figure.

Exam. 2. *Suppoſe the given equation, when prepared for a ſolution, to be* $768z + 48z^2 + z^3 = -96$.

In this caſe $a = 768$, $b = 48$, $c = 1$, $d = 0$, $p = -96$, $q \left(= \frac{p}{a}\right) = -\frac{1}{8}$, $s \left(= \frac{b}{a} - \frac{c}{b}\right) = \frac{1}{16} - \frac{1}{48} = \frac{1}{24}$, and $w \left(= \frac{2b}{a}\right) + \frac{ad - bc}{bb - ac} = \frac{1}{8} - \frac{48}{48 \times 48 - 768} = \frac{1}{8} - \frac{1}{48 - 16} = \frac{3}{32}$. Therefore $z = \frac{p + pqs}{a + \overline{b + as}.q}$

$$= \frac{-96 - 96 \times -\frac{1}{8} \times \frac{1}{24}}{768 + \overline{48 + 32} \times -\frac{1}{8}} = \frac{-96 + \frac{1}{2}}{768 - 6 - 4} = \frac{-191}{1516} =$$

$-0,1259894$, nearly; or $z = \frac{p + pqw}{a + \overline{b + aw}.q + \overline{w - s}.bqq}$

$$= \frac{-96 - 96 \times -\frac{1}{8} \times \frac{3}{32}}{768 + \overline{48 + 72} \times -\frac{1}{8} + \frac{5}{96} \times \frac{48}{64}} = \frac{-96 + \frac{9}{8}}{768 - 6 - 9 + \frac{5}{128}}$$

$$= \frac{-96 \times 128 + 9 \times 16}{753 \times 128 + 5} = -\frac{12144}{96389} = -0,1259894802,$$

more nearly.

In the ſame manner the roots of other equations may be approached: but, to avoid trouble in preparing the equation for a ſolution, you may every-where neglect all ſuch powers of the converging quantity z as would riſe higher than the degree, or order of the approximation you intend to work by. And further to facilitate the labour of ſuch a transformation, the following general equations for the values of p, a, b, c, d, &c. may be uſed.

$p = k - \alpha r - \beta r^2 - \gamma r^3 - \delta r^4$ &c.
$a = \alpha + 2\beta r + 3\gamma r^2 + 4\delta r^3$ &c.
$b = \beta + 3\gamma r + 6\delta r^2 + 10\epsilon r^3$ &c.
$c = \gamma + 4\delta r + 10\epsilon r^2 +$ &c.
$d = \delta + 5\epsilon r +$ &c.

The original equation being $\alpha x + \beta x^2 + \gamma x^3 + \delta x^4 + \epsilon x^5$ &c. $= k$: from whence, by making $r + z = x$, the above values are deduced.

The better to illuſtrate the uſe of what is here laid down, I ſhall ſubjoin another example; wherein let there be given $x^5 + 2x^4 + 3x^3 + 4x^2 + 5x$ (or $5x + 4x^2 + 3x^3 + 2x^4 + x^5$) $= 54321$; to find x, by an approximation of the ſecond degree.

In this caſe, k being $= 54321$, $\alpha = 5$, $\beta = 4$, $\gamma = 3$, $\delta = 2$, and $\varepsilon = 1$, we have

$$p = 54321 - 5r - 4r^2 - 3r^3 - 2r^4 - r^5,$$
$$a = 5 + 8r + 9r^2 + 8r^3 + 5r^4, \text{ and}$$
$$b = 4 + 9r + 12r^2 + 10r^3.$$

Which values, by aſſuming $r = 8$, will become $p = 11529$, $a = 25221$, and $b = 5964$: whence $q \left(= \frac{p}{a}\right) = 0{,}45$, and $z \left(= \frac{p}{a + bq}\right) = \frac{11529}{25221 + 2683} = 0{,}41$; and therefore $x (= r + z) = 8{,}41$, nearly.

To repeat the operation, let $8{,}41$ be now ſubſtituted for r; ſo ſhall $p = 135{,}92$, $a = 30479$, $b = 6876$, $q \left(= \frac{p}{a}\right) = 0.00445$, and $z \left(= \frac{p}{a + bq}\right) = \frac{135{,}92}{30479 + 30} = 0{,}004455$: which added to $8{,}41$, gives $8{,}414455$, for the next value of x.

The *formulæ*, or approximations determined in the preceding pages, are general, anſwering to equations of all degrees, howſoever affected: but in the extraction of the roots of *pure* powers the proceſs will be more ſimple, and the theorems themſelves very much abbreviated.

For let $x^n = k$ be the equation whereof the root x is to be extracted; then, by aſſuming r nearly equal to x, and making $r \times \overline{1 + z} = x$, our equation will become $r^n \times \overline{1 + z}\rvert^n = k$, or $\overline{1 + z}\rvert^n = \frac{k}{r^n}$, that is, $1 + nz + n \cdot \frac{n-1}{2} z^2 + n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot z^3 + n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{n-3}{4} \cdot z^4$ &c. $= \frac{k}{r^n}$: from whence, by tranſpoſition and diviſion, $z + \frac{n-1}{2} \cdot z^2 + \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot z^3$

$+\frac{n-1}{2}.\frac{n-2}{3}.\frac{n-3}{4}.z^4$ &c. $=\frac{k-r^n}{nr^n}$.

Here, by a compariſon with the general equation, $az+bz^2+cz^3+dz^4$ &c. $=p$, we have $a=1$, $b=\frac{n-1}{2}$, $c=\frac{n-1}{2}.\frac{n-2}{3}$, $d=\frac{n-1}{2}.\frac{n-2}{3}.\frac{n-3}{4}$ &c. and $p=\frac{k-r^n}{nr^n}$: whence $q\ (\frac{p}{a})=p$; $s\ (\frac{b}{a}-\frac{c}{b})=\frac{n-1}{2}-\frac{n-2}{3}=\frac{n+1}{6}$; and w

$$\left(\frac{2b}{a}+\frac{\frac{ad}{b}-c}{b-\frac{ac}{b}}\right)=\frac{n-1}{1}+\frac{\frac{1}{12}.\overline{n-2}.\overline{n-3}-\frac{1}{6}.\overline{n-1}.\overline{n-2}}{\frac{1}{2}.\overline{n-1}-\frac{1}{3}.\overline{n-2}}$$

$$=\frac{n-1}{1}+\frac{\overline{n-2}.\overline{n-3}-\overline{2n-2}.\overline{n-2}}{2.\overline{n+1}}=$$

$$\frac{n-1}{1}+\frac{n-2}{2.\overline{n+1}}\times\overline{n-3-2n+2}=\frac{n-1}{1}+$$

$$\frac{n-2}{2.\overline{n+1}}\times-\overline{n+1}=\frac{n-1}{1}-\frac{n-2}{2}=\frac{n}{2}.$$

Therefore, for an approximation of the third degree, we have

$$z=\frac{\overline{a+sp}.p}{aa+\overline{b+as}.p}=\frac{\overline{1+\frac{1}{6}p.\overline{n+1}}.p}{1+\overline{\frac{n-1}{2}+\frac{n+1}{6}}.p}=$$

$$\frac{p+\overline{n+1}.\frac{1}{6}p^2}{1+\overline{2n-1}.\frac{1}{3}p}$$

: and for an approximation of the fourth degree

$$z=\frac{p\times\overline{1+wq}}{a+\overline{b+aw}.q+\overline{w-s}.bq^2}=$$

$$\frac{p+\frac{1}{2}np^2}{1+\overline{\frac{n-1}{2}+\frac{n}{2}}.p+\overline{\frac{n}{2}-\frac{n+1}{6}}.\frac{n-1}{2}.p^2}=$$

$$=$$

$= \dfrac{p + \frac{1}{2}np^2}{1 + \frac{\overline{2n-1}}{2}.p + \frac{2n-1}{2}.\frac{n-1}{6}.p^2}$. Hence it is evident that the root x $(r \times \overline{1+z})$ of the given equation $x^n = k$, will be equal to $r + \dfrac{rp \times \overline{1 + \overline{n+1}.\frac{1}{6}p}}{1 + \overline{2n-1}.\frac{1}{3}p}$, *nearly*; and equal to $r + \dfrac{rp \times \overline{1 + \frac{1}{2}np}}{1 + \frac{2n-1}{2}.p + \frac{\overline{2n-1}.\overline{n-1}}{12}.p^2}$ *more nearly.*

But both theſe theorems will be rendered a little more commodious, by putting $v = \dfrac{nr^n}{k - r^n}$, and ſubſtituting $\dfrac{1}{v}$, in the place of its equal, p, whence, after proper reduction, x will be had $= r + \dfrac{r \times \overline{6v + \overline{n+1}}}{v \times \overline{6v + 4n - 2}}$, *nearly*; and equal to $r + \dfrac{r \times \overline{2v + n}}{v \times \overline{2v + \overline{2n-1}} + \frac{1}{6}.\overline{n-1}.\overline{2n-1}}$, *more nearly.*

I ſhall now put down an example, or two, to ſhew the uſe and great exactneſs of theſe laſt expreſſions.

1. Let the equation given be $x^2 = 2$, or, which is the ſame, let the ſquare root of 2 be required.

Then, aſſuming $r = 1.4$, we have $n = 2$, $k = 2$, v $\left(\dfrac{nr^n}{k - r^n}\right) = \dfrac{2 \times 1{,}96}{2 - 1.96} = 98$; and therefore $r + \dfrac{r \times \overline{6v + \overline{n+1}}}{v \times \overline{6v + 4n - 2}} = 1{,}4 + \dfrac{1{,}4 \times 591}{98 \times 594} = 1{,}4 + \dfrac{197}{70 \times 198} = 1{,}4 + \dfrac{197}{13860} = 1{,}41421356$; which is the value of x according to the former approximation;

but, according to the latter, the anſwer will come out $1.4 + \frac{5544}{39005} = 1.41421356236$; which is true to the laſt figure: and, if with this number the operation be repeated, you will have the anſwer true to nearly 60 places of decimals.

2. Let it be required to extract the cube root of 1728. Here, taking $r = 11$, we ſhall have $v\ (\frac{nr^n}{k - r^n})$ $= \frac{3993}{397} = 10.05793$; and therefore $r +$

$$\frac{r \times \overline{2v + n}}{\overline{2v + 2n - 1} \times v + \frac{1}{6} \times \overline{n - 1} \times \overline{2n - 1}} = 11{,}99998;$$

which differs from truth by only $\frac{1}{50000}$ part of an unit.

3. Let it be propoſed to extract the cube root of 500. Here, the required root appearing to be leſs than 8, but nearer to 8 than 7, let r be taken $= 8$, and we ſhall have $v\ (= \frac{3 \times 512}{-12}) = -128$; and therefore $r +$

$$\frac{r \times \overline{2v + n}}{\overline{2v + 2n - 1} \times v + \frac{1}{6} \times \overline{n - 1} \times \overline{2n - 1}}$$

$= 8 - \frac{6072}{96389} = 7.937005259936$; which number is true to the laſt place.

4. Laſtly, let it be propoſed to extract the firſt ſurſolid root of 125000. In which caſe k being $= 125000$, $n = 5$, $r = 10$, and $v = 20$, the required root will be found $= 10.456389$.

Beſides the different approximations hitherto delivered, there are various other ways whereby the roots of equations may be approached; but, of theſe, none more general, and eaſy in practice, than the following.

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8
64
8
512
500

8
062
7.938

1.92 / 1200 1.62
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Let the general equation, $az + bz^2 + cz^3 + dz^4 + ez^5$ &c. $= p$, be here reſumed; which, by diviſion, becomes $z = \dfrac{1}{\frac{a}{p} + \frac{b}{p}z + \frac{c}{p}z^2 + \frac{d}{p}z^3 + \frac{e}{p}z^4 \text{ \&c.}}$. If, therefore, we make $A = \dfrac{a}{p}$, and neglect all the terms after the firſt, we ſhall have $z = \dfrac{1}{A}$; being an approximation of the firſt degree.

And if this value of z be now ſubſtituted in the ſecond term, and all the following ones be rejected, we ſhall then have $z = \dfrac{1}{\frac{a}{p} + \frac{b}{p} \times \frac{1}{A}} = \dfrac{A}{\frac{a}{p}A + \frac{b}{p}} = \dfrac{A}{B}$ (by making $B = \dfrac{aA + b}{p}$); which is an approximation of the ſecond degree.

In order now to get an approximation of the third degree, let this laſt value be ſubſtituted in the ſecond term, neglecting all the terms after the third; ſo ſhall $z = \dfrac{1}{\frac{a}{p} + \frac{b}{p} \times \frac{A}{B} + \frac{c}{p}z^2}$: but here, in the room of z^2, either of the ſquares of the two preceding values of z, or their rectangle may be ſubſtituted, that is, either $\dfrac{1}{A} \times \dfrac{1}{A}$, $\dfrac{A}{B} \times \dfrac{A}{B}$, or $\dfrac{1}{A} \times \dfrac{A}{B}$; but the laſt of theſe $= \left(\dfrac{1}{B}\right)$ is the moſt commodious; whence we have $z = \dfrac{B}{\frac{a}{p}B + \frac{b}{p}A + \frac{c}{p}} = \dfrac{B}{C}$; ſuppoſing $C = \dfrac{aB + bA + c}{p}$.

Again, for an approximation of the fourth degree, we have $\dfrac{b}{p}z = \dfrac{b}{p} \times \dfrac{B}{C}$; $\dfrac{c}{p}z^2 = \dfrac{c}{p} \times \dfrac{B}{C} \times \dfrac{A}{B} = \dfrac{c}{p} \times \dfrac{A}{C}$; and $\dfrac{d}{p}z^3 = \dfrac{d}{p} \times \dfrac{B}{C} \times \dfrac{A}{B} \times \dfrac{1}{A} = \dfrac{d}{p} \times \dfrac{1}{C}$; which values

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3269 /363/
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121
121
1331

397
44
353

1
1
4
4
11
44

but, according to the latter, the anſwer will come out $1.4 + \frac{5544}{39005} = 1.41421356236$; which is true to the laſt figure: and, if with this number the operation be repeated, you will have the anſwer true to nearly 60 places of decimals.

2. Let it be required to extract the cube root of 1728. Here, taking $r = 11$, we ſhall have $v \left(\frac{nr^n}{k - r^n}\right) = \frac{3993}{397} = 10.05793$; and therefore $r + \frac{r \times \overline{2v + n}}{\overline{2v + 2n - 1} \times v + \frac{1}{6} \times \overline{n - 1} \times \overline{2n - 1}} = 11,99998$; which differs from truth by only $\frac{1}{50000}$ part of an unit.

3. Let it be propoſed to extract the cube root of 500. Here, the required root appearing to be leſs than 8, but nearer to 8 than 7, let r be taken $= 8$, and we ſhall have $v \left(= \frac{3 \times 512}{-12}\right) = -128$; and therefore $r + \frac{r \times \overline{2v + n}}{\overline{2v + 2n - 1} \times v + \frac{1}{6} \times \overline{n - 1} \times \overline{2n - 1}} = 8 - \frac{6072}{96389} = 7.937005259936$; which number is true to the laſt place.

4. Laſtly, let it be propoſed to extract the firſt ſurſolid root of 125000. In which caſe k being $= 125000$, $n = 5$, $r = 10$, and $v = 26$, the required root will be found $= 10.456389$.

Beſides the different approximations hitherto delivered, there are various other ways whereby the roots of equations may be approached; but, of theſe, none more general, and eaſy in practice, than the following.

11

8
062
7.938

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64
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192/

1.62

Let the general equation, $ax + bx^2 + cx^3 + dx^4 + ex^5$ &c. $= p$, be here reſumed; which, by diviſion, becomes $x = \frac{1}{\frac{a}{p} + \frac{b}{p}x + \frac{c}{p}x^2 + \frac{d}{p}x^3 + \frac{e}{p}x^4 \,\&c.}$. If, therefore, we make $A = \frac{a}{p}$, and neglect all the terms after the firſt, we ſhall have $x = \frac{1}{A}$; being an approximation of the firſt degree.

And if this value of x be now ſubſtituted in the ſecond term, and all the following ones be rejected, we ſhall then have $x = \frac{1}{\frac{a}{p} + \frac{b}{p} \times \frac{1}{A}} = \frac{A}{\frac{a}{p}A + \frac{b}{p}} = \frac{A}{B}$ (by making $B = \frac{aA + b}{p}$); which is an approximation of the ſecond degree.

In order now to get an approximation of the third degree, let this laſt value be ſubſtituted in the ſecond term, neglecting all the terms after the third; ſo ſhall $x = \frac{1}{\frac{a}{p} + \frac{b}{p} \times \frac{A}{B} + \frac{c}{p}x^2}$: but here, in the room of x^2, either of the ſquares of the two preceding values of x, or their rectangle may be ſubſtituted, that is, either $\frac{1}{A} \times \frac{1}{A}$, $\frac{A}{B} \times \frac{A}{B}$, or $\frac{1}{A} \times \frac{A}{B}$; but the laſt of theſe $= (\frac{1}{B})$ is the moſt commodious; whence we have $x = \frac{B}{\frac{a}{p}B + \frac{b}{p}A + \frac{c}{p}} = \frac{B}{C}$; ſuppoſing $C = \frac{aB + bA + c}{p}$.

Again, for an approximation of the fourth degree, we have $\frac{b}{p}x = \frac{b}{p} \times \frac{B}{C}$; $\frac{c}{p}x^2 = \frac{c}{p} \times \frac{B}{C} \times \frac{A}{B} = \frac{c}{p} \times \frac{A}{C}$; and $\frac{d}{p}x^3 = \frac{d}{p} \times \frac{B}{C} \times \frac{A}{B} \times \frac{1}{A} = \frac{d}{p} \times \frac{1}{C}$; which values

3530(97
3269 97/363)
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1728
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397 /1.09
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3400
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121
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397
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lues being ſubſtituted in the general equation and all the terms after the four firſt rejected, there now comes out

$$z = \frac{1}{\frac{a}{p} + \frac{bB}{pC} + \frac{cA}{pC} + \frac{d}{pC}} = \frac{C}{\frac{a}{p}C + \frac{b}{p}B + \frac{c}{p}A + \frac{d}{p}}$$

$= \frac{C}{D}$; by making $D = \frac{aC + bB + cA + d}{p}$.

In like manner, for an approximation of the fifth degree, we ſhall have $\frac{b}{p}z = \frac{b}{p} \times \frac{C}{D}$, $\frac{c}{p}z^2 = \frac{c}{p} \times \frac{C}{D} \times \frac{B}{C} = \frac{cB}{pD}$, $\frac{d}{p}z^3 = \frac{d}{p} \times \frac{C}{D} \times \frac{B}{C} \times \frac{A}{B} = \frac{dA}{pD}$, and $\frac{e}{p}z^4 = \frac{C}{D} \times \frac{B}{C} \times \frac{A}{B} \times \frac{1}{A} = \frac{e}{pD}$; and conſequently z

$$= \frac{D}{\frac{a}{p}D + \frac{b}{p}C + \frac{c}{p}B + \frac{d}{p}A + \frac{e}{p}} = \frac{D}{E};$$ ſuppoſing

$E = \frac{aD + bC + cB + dA + e}{p}$. Whence the law of continuation is manifeſt; whereby it appears, *that* if there be taken $A = \frac{a}{p}$, $B = \frac{aA + b}{p}$, $C = \frac{aB + bA + c}{p}$,

$D = \frac{aC + bB + cA + d}{p}$, $E = \frac{aD + bC + cB + dA + e}{p}$,

$F = \frac{aE + bD + cC + dB + eA + f}{p}$, $G = \frac{aF + bE + cD + dC + eB + fA + g}{p}$,

&c. then will $\frac{1}{A}$, $\frac{A}{B}$, $\frac{B}{C}$, $\frac{C}{D}$, $\frac{D}{E}$, $\frac{E}{F}$, $\frac{F}{G}$, &c. be ſo many, ſucceſſive, approximations to the value of z, aſcending gradually, from the loweſt to the ſuperior orders.

An example will help to explain the uſe of what is above delivered; wherein we will ſuppoſe the equation given to be $12z + 6z^2 + z^3 = 2$.

Here $a = 12$, $b = 6$, $c = 1$, $d = 0$, $e = 0$, &c. and $p = 2$; whence $A\ (= \frac{a}{p}) = 6$, $B\ (= \frac{aA + b}{p}) = \frac{12 \times 6 + 6}{2}$

$= 39$, $C\ (= \frac{aB + bA + c}{p}) = \frac{12 \times 39 + 6 \times 6 + 1}{2} =$

$= \frac{505}{2}$, D $(= \frac{aC + bB + cA + d}{p}) = \frac{6\times505+6\times39+6}{2}$ $= 1635$, &c.

Therefore, $\frac{A}{B} = \frac{2}{13} = z$, *nearly.*

$\frac{B}{C} = \frac{78}{505} = z$, *more nearly.*

$\frac{C}{D} = \frac{101}{654} = z$, *ſtill nearer.*

From the ſame equations the general values of B, C, D, &c. may be eaſily found, in known terms, independent of each other.

Thus B $(= \frac{aA}{p} + \frac{b}{p}) = \frac{a^2}{p^2} + \frac{b}{p}$ (becauſe A $= \frac{a}{p}$); alſo C $(= \frac{aB}{p} + \frac{bA}{p} + \frac{c}{p}) = \frac{a^3}{p^3} + \frac{2ab}{p^2} + \frac{c}{p}$; and D $(= \frac{aC}{p} + \frac{bB}{p} + \frac{cA}{p} + \frac{d}{p}) = \frac{a^4}{p^4} + \frac{3a^2b}{p^3} + \frac{2ac+bb}{p^2} + \frac{d}{p}$ &c. Therefore

$$\frac{A}{B} = \frac{ap}{a^2 + bp};$$

$$\frac{B}{C} = \frac{p \times \overline{a^2 + bp}}{a^3 + 2abp + cp^2};$$

$$\frac{C}{D} = \frac{p \times \overline{a^3 + 2abp + cp^2}}{a^4 + 3a^2bp + \overline{2ac + bb}\,.\,p^2 + dp^3};$$

$$\frac{D}{E} = \frac{p \times \overline{a^4 + 3a^2bp + \overline{2ac + bb}\,.\,p^2 + dp^3}}{a^5 + 4a^3bp + \overline{3ac + 3bb} : ap^2 + \overline{bc + ad}\,.\,2p^3 + ep^4},$$

&c. which are ſo many different approximations to the value of z.

Thus far regard has been had to equations which conſiſt of the ſimple powers of one unknown quantity, and are no-ways affected, either by ſurds or fractions. If either of theſe kinds of quantities be concerned in an equation, the uſual way is to exterminate them by multiplication, or involution (as has been taught in

Sect. IX.) But as this method is, in many caſes, very laborious, and in others altogether impracticable, eſpecially, where ſeveral ſurds are concerned in the ſame equation, it may not be amiſs to ſhew how the method of converging ſeries's may be alſo extended to theſe caſes, without any ſuch previous reduction. In order to which it will be neceſſary to premiſe, that if $A + B$ repreſents a compound quantity, conſiſting of two terms, and the latter (B) be but ſmall in compariſon of the former; then will,

$$\left.\begin{array}{l}
1°.\ \dfrac{1}{A+B} = \dfrac{1}{A} - \dfrac{B}{A^2} \text{ or } \dfrac{1}{A} \times \overline{1 - \dfrac{B}{A}} \\[2ex]
2°.\ \overline{A+B}|^{\frac{1}{2}} = A^{\frac{1}{2}} + \dfrac{B}{2A^{\frac{1}{2}}} \text{ or } A^{\frac{1}{2}} + \dfrac{A^{\frac{1}{2}}B}{2A} \\[2ex]
3°.\ \dfrac{1}{\overline{A+B}|^{\frac{1}{2}}} = \dfrac{1}{A^{\frac{1}{2}}} - \dfrac{B}{2A^{\frac{3}{2}}} \text{ or } \dfrac{1}{A^{\frac{1}{2}}} - \dfrac{B}{2A \times A^{\frac{1}{2}}} \\[2ex]
4°.\ \overline{A+B}|^{\frac{1}{3}} = A^{\frac{1}{3}} + \dfrac{B}{3A^{\frac{2}{3}}} \text{ or } A^{\frac{1}{3}} + \dfrac{A^{\frac{1}{3}}B}{3A} \\[2ex]
5°.\ \dfrac{1}{\overline{A+B}|^{\frac{1}{3}}} = \dfrac{1}{A^{\frac{1}{3}}} - \dfrac{B}{3A^{\frac{4}{3}}} \text{ or } \dfrac{1}{A^{\frac{1}{3}}} - \dfrac{B}{3A \times A^{\frac{1}{3}}} \\[2ex]
6°.\ \overline{A+B}|^{\frac{1}{4}} = A^{\frac{1}{4}} + \dfrac{B}{4A^{\frac{3}{4}}} \text{ or } A^{\frac{1}{4}} + \dfrac{BA^{\frac{1}{4}}}{4A} \\[2ex]
7°.\ \dfrac{1}{\overline{A+B}|^{\frac{1}{4}}} = \dfrac{1}{A^{\frac{1}{4}}} - \dfrac{B}{4A^{\frac{5}{4}}} \text{ or } \dfrac{1}{A^{\frac{1}{4}}} - \dfrac{B}{4A \times A^{\frac{1}{4}}}
\end{array}\right\}\text{, nearly.}$$

All which will appear evident from the general theorem at p. 41: from whence theſe particular equations, or theorems, may be continued at pleaſure; the values here exhibited being nothing more than the two firſt terms of the ſeries there given. But now, to apply them to the purpoſe above mentioned, let there be given $\sqrt{1 + x^2} + \sqrt{2 + x^2} + \sqrt{3 + x^2} = 10$, as an example, where, x being about 3, let $3 + e$ be therefore ſubſtituted for x, rejecting all the powers of e above the firſt, as inconſiderable, and then the given equation will ſtand thus, $\sqrt{10+6e}+\sqrt{11+6e}+\sqrt{12+6e}=10$: but, *by Theorem* 2, $\sqrt{10+6e}$ will be $=\sqrt{10}+\frac{3\sqrt{10}\times e}{10}$, nearly; for, in this caſe, $A=10$, and $B=6e$, and therefore $A^{\frac{1}{2}}+\frac{A^{\frac{1}{2}}B}{2A}=\sqrt{10}+\frac{3\sqrt{10}\times e}{10}$: in like manner is $\sqrt{11+6e}=\sqrt{11}+\frac{3\sqrt{11}\times e}{11}$, &c. and conſequently $\sqrt{10}+\frac{3\sqrt{10}\times e}{10}+\sqrt{11}+\frac{3\sqrt{11}\times e}{11}+\sqrt{12}+\frac{3\sqrt{12}\times e}{12}=10$; which, contracted, gives $9.944+2.718e=10$; whence $2.718e=.056$ and $e=.0205$; conſequently $x=3.0205$, nearly. Wherefore, to repeat the operation, let $3.0205+e$ be now ſubſtituted for x; then will $\sqrt{10.12342+6.041e}+\sqrt{11.12342+6.041e}+\sqrt{12.12342+6.041e}=10$; whence, *by Theorem* 2, $\sqrt{10.12342}+\frac{6.041e}{2\sqrt{10.12342}}+\sqrt{11.12342}+\frac{6.041e}{2\sqrt{11.12342}}+\sqrt{12.12342}+\frac{6.041e}{2\sqrt{12.12342}}=10$, or $9.9987814+2.7224e=10$: from which e comes out $=.000447$, and therefore $x=3.020947$; which is true to the laſt place.

Again, let it be propoſed to find the root of the equation $\frac{20x}{\sqrt{16+5x+x^2}}+\frac{x\sqrt{5+x^2}}{25}=34$. Put $20+e=x$; then, by proceeding as before, we ſhall have $\frac{400+20e}{\sqrt{516+45e}}+\frac{\overline{20+e}\times\sqrt{405+40e}}{25}=34$: but (*by Theorem* 3) $\frac{1}{\sqrt{516+45e}}$ is nearly $=\frac{1}{\sqrt{516}}-$

45e

$\frac{45e}{1032 \times \sqrt{516}}$, and *(by Theorem 2)* $\sqrt{\overline{405 + 40e}} = \sqrt{405} + \frac{20e}{\sqrt{405}}$: which values being ſubſtituted above, our equation becomes

$$\overline{400 + 20e} \times \overline{\frac{1}{\sqrt{516}} - \frac{45e}{1032 \times \sqrt{516}}} + \frac{20+e}{25} \times \overline{\sqrt{405} + \frac{20e}{\sqrt{405}}}$$

$= 34$, that is, $\overline{400 + 20e} \times \overline{.044022 - .00192e} + \overline{20 + e} \times \overline{.804984 + .0398e} = 34$; whence rejecting e^2, &c. we have $1.713e = .1915$; and conſequently $e = .1118$.

Thirdly, let there be given $\sqrt{\overline{1 - x}} + \sqrt{\overline{1 - 2x^2}} + \sqrt{\overline{1 - 3x^3}} = 2$. Then, if $0.5 + e$ be ſubſtituted therein for x, *it* will become $\sqrt{\overline{0.5 - e}} + \sqrt{\overline{0.5 - 2e}} + \sqrt{\overline{0.625 - 2.25e}} = 2$; or $\sqrt{0.5} - \sqrt{0.5} \times e + \sqrt{0.5} - \sqrt{0.5} \times 2e + \sqrt{.625} - \frac{2{,}25e}{2\sqrt{.625}} = 2$; whence $3.545e = .204$, $e = .057$, and $x = 0.557$, with which the operation being repeated, the next value of x will come out $= .5516$.

Laſtly, let there be given $\overline{1 + x}|^{\frac{1}{2}} + \overline{1 + x^2}|^{\frac{1}{3}} + \overline{1 + x^3}|^{\frac{1}{4}} = 6{,}5$. Here, by writing $3 + e$ for x, and proceeding as above, we ſhall have $2 + \frac{e}{4} + \overline{10}|^{\frac{1}{3}} + \frac{\overline{10}|^{\frac{1}{3}} \times 2e}{10} + \overline{28}|^{\frac{1}{4}} + \frac{\overline{28}|^{\frac{1}{4}} \times 27e}{4 \times 28} = 6.5$, that is, $6.455 + 1.23e = 6.5$; whence $e = .036$, and $x = 3.036$.

It may be obſerved that this method, as all the powers of e above the firſt are rejected, only doubles the number of places, at each operation: but, from what is therein ſhewn, it is eaſy to ſee how it may be extended, ſo as to triple, or even quadruple, that number; but then the trouble, in every operation, would be increaſed in proportion, ſo that little, or no advantage could be reaped therefrom.

Hitherto we have treated of equations which include one unknown quantity, only. If there be two equations given, and as many quantities (x and y) to be determined, one of thoſe quantities muſt firſt be exterminated, and the two equations reduced to one, according to what is ſhewn in Sect. 9. But, if this cannot be readily done (which is ſometimes the caſe) and the unknown quantities be ſo entangled as to render that way impracticable, the following method may be of uſe.

Let the values of x and y be aſſumed pretty near the truth (which, from the nature of the problem, may always be done); and let the values ſo aſſumed be denoted by f, and g, and what they want of truth by s, and t reſpectively; that is, let $f + s = x$, and $g + t = y$: ſubſtitute theſe values in both equations, rejecting (by reaſon of their ſmallneſs) all the terms wherein more than one ſingle dimenſion of the quantities s and t are concerned: let all the terms in the firſt equation, which are affected by s, be collected under their proper ſigns, and denoted by As; in like manner, let thoſe affected by t, be denoted by Bt; and thoſe affected neither by s nor t, by Q: moreover, let the terms of the ſecond equation, wherein s and t are concerned, be denoted by as, and bt, reſpectively; and let the known terms, on the right-hand ſide of this equation, or thoſe in which neither s, nor t enters, be repreſented by q. Then the equations (be they of what kind they will) will ſtand thus, $As + Bt = Q$, and $as + bt = q$. By multiplying the former of which by b, and the latter by B, and then ſubtracting the one from the other, we ſhall have $bAs - Bas = bQ - Bq$; and therefore $s = \frac{bQ - Bq}{Ab - aB}$; whence $x\ (= f + s)$ is given.

Again, by multiplying the former equation by a, and the latter by A, &c. we ſhall have $aBt - Abt = aQ - Aq$, and therefore $t = \frac{aQ - Aq}{Ba - bA} = \frac{Aq - aQ}{Ab - aB}$: whence $y\ (= g + t)$ is likewiſe given.

It is eaſy to ſee that this method is alſo applicable, in caſe of three, or four equations, and as many unknown quantities, but as theſe are caſes that ſeldom occur in the reſolution of problems; and, when they do, are reducible to thoſe already conſidered, it will be needleſs to take further notice of them here: I ſhall, therefore, content myſelf with giving an example, or two, of the uſe of what is above laid down.

1. Let there be given $x^4 + y^4 = 10000$, and $x^5 - y^5 = 25000$; to find x and y. Then, by writing $f + s = x$, $g + t = y$, and proceeding according to the aforegoing directions, we ſhall have $f^4 + 4f^3s + g^4 + 4g^3t = 10000$, and $f^5 + 5f^4s - g^5 - 5g^4t = 25000$, or $4f^3s + 4g^3t = 10000 - f^4 - g^4$, and $5f^4s - 5g^4t = 25000 + g^5 - f^5$: therefore, in this caſe, $A = 4f^3$, $B = 4g^3$, $Q = 10000 - f^4 - g^4$, $a = 5f^4$, $b = -5g^4$, and $q = 25000 + g^5 - f^5$. But it appears, from the firſt of the two given equations, that x muſt be ſomething leſs than 10, and from the ſecond, that y muſt be leſs than x: I therefore take $f = 9$, and $g = 8$; and then A becomes $= 2916$, $B = 2048$, $Q = -657$, $a = 32805$, $b = -20480$, $q = -1281$; and therefore $s\,(\frac{bQ - Bq}{bA - Ba}) = -0.13$, and $t\,(\frac{Aq - aQ}{bA - Ba}) = -0.14$; hence $x = 8.87$, and $y = 7.86$, nearly.

Therefore, in order to repeat the operation, let f be now taken $= 8.87$, and $g = 7.86$; then will $A = 2791$, $B = 1942$, $Q = -6.76$, $a = 30950$, $b = -19083$, and $q = 94$; conſequently $s\,(= \frac{bQ - Bq}{Ab - aB}) = .00047$, and $t\,(= \frac{Aq - aQ}{Ab - aB}) = -.00415$; whence $x = 8.87047$, and $y = 7.85585$; both which values are true to the laſt figure.

Example 2. Let there be given $\overline{20x + xy^2}\big|^{\frac{1}{3}} + \overline{8x}\big|^{\frac{1}{2}} = 12$, and $\sqrt{x^2 + y^2} + \frac{xy}{\sqrt{x^2 - y^2}} = 13$. Here the given equations, by writing $f + s$ for x, and $g + t$

for y will become $\overline{20f+20s+fg^2+2fgt+g^2s}|^{\frac{1}{3}} + \sqrt{8f+8s} = 12$, and $\sqrt{f^2+g^2+2fs+2gt} + \frac{fg+ft+gs}{\sqrt{f^2-g^2+2fs-2gt}} = 13$: but $\overline{20f+fg^2+20s+2fgt+g^2s}|^{\frac{1}{3}}$, by what is ſhewn in p. 174, will be transformed to $\overline{20f+fg^2}|^{\frac{1}{3}} + \frac{\overline{20f+fg^2}|^{\frac{1}{3}}}{3\times\overline{20f+fg^2}} \times \overline{20s+2fgt+g^2s}$ (ſuppoſing all the terms that have more than one dimenſion of s and t, to be rejected, as inconſiderable); alſo $\sqrt{f^2+g^2+2fs+2gt}$, is tranſformed to $\sqrt{f^2+g^2} + \frac{fs+gt}{\sqrt{f^2+g^2}}$, and $\frac{1}{\sqrt{f^2-g^2+2fs-2gt}}$ to $\frac{1}{\sqrt{f^2-g^2}} - \frac{fs-gt}{\overline{f^2-g^2}\times\sqrt{f^2-g^2}}$; therefore our equations will ſtand thus,

$$\overline{20f+fg^2}|^{\frac{1}{3}} + \frac{\overline{20f+fg^2}|^{\frac{1}{3}} \times \overline{20s+2fgt+g^2s}}{3\times\overline{20f+fg^2}} + \sqrt{8f} + \frac{4s}{\sqrt{8f}} = 12,$$

and

$$\sqrt{f^2+g^2} + \frac{fs+gt}{\sqrt{f^2+g^2}} + \overline{fg+ft+gs} \times \overline{\frac{1}{\sqrt{f^2-g^2}} - \frac{fs-gt}{\overline{f^2-g^2}\times\sqrt{f^2-g^2}}} = 13:$$

which equations, if f be aſſumed $= 5$, and $g = 4$, will be reduced to $5.6462 + .01045 \times \overline{36s+40t} + 6.3245 + .6324s = 12$, and $6.4031 + .781s + .625t + \overline{20+5t+4s} \times \overline{.3333-.1852s+.1482t} = 13$; whence $1.008s + .418t = .0293$, and $1.59s - 5.255t = .0698$: therefore, in this caſe, $A = 1.008$, $B = 0.418$, $Q = .0293$, $a = 1.59$, $b = -5.255$, and $q = .0698$: conſequently s $(= \frac{bQ-Bq}{Ab-aB}) = 0.305$, and t $(= \frac{Aq-aQ}{Ab-aB}) = -.0040$; therefore $x = 5.0305$, and $y = 3.9960$.

SECTION XIII.

Of indeterminate, or unlimited Problems.

A Problem is ſaid to be indeterminate, or unlimited, when the equations, expreſſing the conditions thereof, are fewer in number than the unknown quantities to be determined; ſuch kinds of Problems, ſtrictly ſpeaking, being capable of innumerable anſwers: but the anſwers in whole numbers, to which the queſtion is commonly reſtrained, are, for the general part, limited to a determinate number; for the more ready diſcovering of which, I ſhall premiſe the following

LEMMA.

Suppoſing $\frac{ax \pm b}{c}$ to be an algebraic fraction, in it's loweſt terms, x being indeterminate, and a, b, and c given whole numbers; then, I ſay, that the leaſt integer, for the value of x, that will alſo give the value of $\frac{ax \pm b}{c}$ an integer, will be found by the following method of calculation.

Divide the denominator (c) by the coefficient (a) of the indeterminate quantity; alſo divide the diviſor by the remainder, and the laſt diviſor, again, by the laſt remainder; and ſo on, till an unit only remains.

Write down all the quotients in a line, as they follow; under the firſt of which write an unit, and under the ſecond write the firſt; than multiply theſe two together, and having added the firſt term of the lower line (or an unit) to the product, place the ſum under the third term of the upper line: multiply, in like manner, the next two correſponding terms of the two lines together, and, having added the ſecond term of the lower *to the product, put down the reſult under the fourth term of the upper one: proceed on, in this way, till you have multiplied by every number in the upper line.*

Then multiply the laſt number thus found by the abſolute quantity (b) in the numerator of the given fraction, and divide the product by the denominator; ſo ſhall the remainder be the true value of x, required; provided the number of terms in the upper line be even and the ſign of b negative, or, if that number be odd and the ſign of b affirmative; but, if the number of terms be even and the ſign of b affirmative, or vice verſâ; *then the difference between the ſaid remainder and the denominator of the fraction will be the true anſwer.*

In the general method here laid down a is ſuppoſed leſs than c, and that theſe two numbers are *prime* to each other: for, were they to admit of a common meaſure, whereby b is not diviſible, the thing would be impoſſible, that is, no integer could be aſſigned for x, ſo as to give the value of $\frac{ax \pm b}{c}$ an integer: the reaſon of which, as well as of the *lemma* itſelf, will be explained a little farther on: here it will be proper to put down an example or two, to illuſtrate the uſe of what has been already delivered.

Examp. 1. *Let the quantity given be* $\frac{87x - 50}{256}$.

Then the operation will ſtand as follows.

```
87)256(2
   82)87(1          2, 1, 16,  2
       5)82(16      1, 2,  3, 50,  103
          2)5(2                     50
            1               256)5150(20
                                  30 = x.
```

Examp. 2. *Given* $\frac{71x + 10}{89}$.

```
71)89(1
   18)71(3          1, 3, 1
      17)18(1       1, 1, 4, 5
          1                  10
                             50 = x.
```

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Write down all the quotients in a line, as they follow; under the firſt of which write an unit, and under the ſecond write the firſt; then multiply theſe two together, and having added the firſt term of the lower line (or an unit) to the product, place the ſum under the third term of the upper line: multiply, in like manner, the next two correſponding terms of the two lines together, and, having added the ſecond term of the lower *to the product, put down the reſult under the fourth term of the upper one: proceed on, in this way, till you have multiplied by every number in the upper line.*

Then multiply the laſt number thus found by the abſolute quantity (b) in the numerator of the given fraction, and divide the product by the denominator; ſo ſhall the remainder be the true value of x, required; provided the number of terms in the upper line be even and the ſign of b negative, or, if that number be odd and the ſign of b affirmative; but, if the number of terms be even and the ſign of b affirmative, or vice verſâ; *then the difference between the ſaid remainder and the denominator of the fraction will be the true anſwer.*

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```
87)256(2
   82)87(1          2, 1, 16, 2
       5)82(16      1, 2, 3, 50, 103
          2)5(2                   50
            1             256)5150(20
                                 30 = x.
```

Examp. 2. *Given* $\frac{71x + 10}{89}$.

```
71)89(1
   18)71(3          1, 3, 1
      17)18(1       1, 1, 4, 5
           1                10
                            50 = x.
```

Examp. 3. *Given* $\frac{377x - 250}{450}$.

377)450(1
73)377(5
12)73(6
1

1, 5, 6
1, 1, 6, 37
250
1850
74
450)9250(20
250
450
200 = x.

Examp. 4. *Given* $\frac{987x + 651}{1235}$.

987)1235(1
248)987(3
243)248(1
5)243(48
3)5(1
2)3(1
1

1, 3, 1, 48, 1, 1
1, 1, 4, 5, 244, 249, 493
651
493
2465
2958
1235)320943(259
7394
12193
1078
1235
157 = x.

Theſe four examples comprehend all the different caſes that can happen with regard to the reſtrictions ſpecified in the latter part of the rule: I ſhall now ſhew the uſe thereof in the reſolution of problems.

PROBLEM I.

To find the leaſt whole number, which, divided by 17, *ſhall leave a remainder of* 7; *but being divided by* 26, *the remainder ſhall be* 13.

Let x be the quotient, by 17, when 7 remains, or, which is the ſame, let $17x + 7$ expreſs the number ſought; then, ſince this number, when 13 is ſubtracted from it, is diviſible by 26, it is manifeſt that $\frac{17x + 7 - 13}{26}$, or $\frac{17x - 6}{26}$ muſt be a whole number: whence, by proceeding according to the *lemma*, x will be found $= 8$; and conſequently $17x + 7 = 143$, the number required. See the operation.

```
17)26(1
   17
   --
    9)17(1          1, 1, 1,
       9            1; 1, 2, 3
      --                     6
       8)9(1                --
         9                  18
         -                  26
         1                  --
                             8 = x.
```

PROBLEM II.

Suppoſing $9x + 13y = 2000$, *it is required to find all the poſſible values of* x *and* y *in whole poſitive numbers.*

By tranſpoſing $13y$, and dividing the whole equation by 9, we have $x = \frac{2000 - 13y}{9} = 222 - y + \frac{2 - 4y}{9}$; which, as x is a whole poſitive number, by the queſtion, muſt alſo be a whole poſitive number, and ſo likewiſe $\frac{4y - 2}{9}$; from which the leaſt value of y, in whole numbers, will come out $= 5$; and conſequently the correſponding value of $x = 215$. From whence the reſt of the anſwers, which are 16 in number, will be found, by adding 9, continually to the laſt value of y, and ſubtracting 13 from that of x, as in the annexed table, which exhibits all the poſſible anſwers in whole numbers,

$x =$	215	202	189	176	163	150	137	124	111	98	85	72	59	46	33	20	7
$y =$	5	14	23	32	41	50	59	68	77	86	95	104	113	122	131	140	149

In the ſame manner, the leaſt value of y, and the greateſt of x being found, in any other caſe, the reſt of the anſwers will be obtained, by only adding the coefficient of x, in the given equation, to the laſt value of y, *continually*, and ſubtracting the coefficient of y from the correſponding value of x. Hence it follows, that, if the *greateſt* value of x be divided by the coefficient of y, the remainder will be the *leaſt* value of x, and that the quotient + 1 will give the number of all the anſwers. But it is to be obſerved, that the equations here ſpoken of, are ſuch, wherein the ſaid coefficients are prime to each other; if this ſhould not be the caſe, let the equation given be, firſt of all, reduced to one of this form, by dividing by the greateſt common meaſure.

PROBLEM III.

To find how many different ways it is poſſible to pay 100 l. *in guineas and piſtoles, only; reckoning guineas at* 21 *ſhillings each, and piſtoles at* 17.

Let x repreſent the number of guineas, and y that of the piſtoles; then, the number of ſhillings in the guineas being $21x$, and in the piſtoles, $17y$, we ſhall therefore have $21x + 17y = 2000$, and conſequently $x = \frac{2000 - 17y}{21} = 95 + \frac{5 - 17y}{21}$; which being a whole number, by the queſtion, it is manifeſt that $\frac{17y - 5}{21}$ muſt alſo be an integer; now the leaſt value of y, in whole numbers, to anſwer this condition, will be found $= 4$, and the expreſſion itſelf $= 3$; the correſponding, or greateſt value of x being $= 92$; which being divided by 17, the coefficient of y (according to the preceding *note)* the quotient comes out 5, and the remainder 7: therefore the leaſt value of x is 7, and the number of anſwers $(= 5 + 1) = 6$; and theſe are as follow,

$x =$	92	75	58	41	24	7.
$y =$	4	25	46	67	88	109.

PROBLEM IV.

To determine whether it be poſſible to pay 100 l. *in guineas and moidores* only; *the former being reckoned at* 21 *ſhillings each, and the latter at* 27.

Here, by proceeding as in the laſt queſtion, we have

$21x + 27y = 2000$; and conſequently $x = \frac{2000 - 27y}{21}$ $= 95 - y - \frac{6y - 5}{21}$: where, the fraction being in its leaſt terms, and the numbers 6 and 21, at the ſame time, admitting of a common meaſure, a ſolution in whole numbers *(by the note to the preceding lemma)* is impoſſible. The reaſon of which depends on theſe two conſiderations; that, whatſoever number is diviſible by a given number, muſt be diviſible alſo by all the diviſors of *it*; and that any quantity which exactly meaſures the whole and one part of another, muſt do the like by the remaining part. Thus, in the preſent caſe, the quantity $6y - 5$, to have the reſult a whole number, ought to be diviſible by 21, and therefore diviſible by 3, likewiſe (which is, here, a common meaſure of a and c): but $6y$, the former part of $6y - 5$, is diviſible by 3, therefore the latter part $- 5$ ought alſo to be diviſible by 3; which is not the caſe, and ſhews the thing propoſed to be impoſſible.

PROBLEM V.

A butcher bought a certain number of ſheep and oxen, for which he paid 100 *l.*; *for the ſheep he paid* 17 *ſhillings apiece, and for the oxen, one with another, he paid* 7 *pounds apiece; it is required to find how many he had of each ſort.*

Let x be the number of ſheep, and y that of the oxen; then, the conditions of the queſtion being expreſſed in algebraic terms, we ſhall have this equation, *viz.* $17x + 140y = 2000$; and conſequently $x = \frac{2000 - 140y}{17} = 117 - 8y - \frac{4y - 11}{17}$; which being a whole number, $\frac{4y - 11}{17}$ muſt therefore be a whole number likewiſe: whence, by proceeding as above, we find $y = 7$, and $x = 60$; and this is the only anſwer the queſtion will admit of: for the greateſt value of x, cannot in this caſe be divided by the coefficient of y, that is 140 cannot be had in 60; and therefore, according to the preceding note, the queſtion can have only one anſwer, in whole numbers.

PROBLEM VI.

A certain number of men and women being merry-making together, the reckoning came to 33 *ſhillings, towards the diſcharging of which, each man paid* 3s. 6d. *and each woman* 1s. 4d.: *the queſtion is, to find how many perſons of both ſexes the company conſiſted of.*

Let x repreſent the number of men, and y that of the women; ſo ſhall $42x + 16y = 396$, or $21x + 8y = 198$; and conſequently $y = \frac{198 - 21x}{8} = 24 - 2x - \frac{5x - 6}{8}$: whence, y being a whole number, $\frac{5x - 6}{8}$ muſt likewiſe be a whole number; and the value of x, anſwering this condition, will be found $= 6$; and conſequently that of y $(= 24 - 12 - 3) = 9$; which two will appear to be the only numbers that can anſwer the conditions of the queſtion; becauſe 21, the coefficient of x, is here greater than 9, the greateſt value of y.

PROBLEM VII.

One bought 12 *loaves for* 12 *pence, whereof ſome were two-penny ones, others penny ones, and the reſt farthing ones: what number were there of each ſort?*

Put $x =$ the number of the firſt ſort, $y =$ that of the ſecond, and $z =$ that of the third; and then, by the conditions of the queſtion, we have theſe two equations, *viz.*

$$x + y + z = 12, \text{ and}$$
$$8x + 4y + z = 48.$$

Whereof the former being ſubtracted from the latter, in order to exterminate z, we thence get $7x + 3y = 36$, and therefore $y = \frac{36 - 7x}{3} = 12 - 2x - \frac{x}{3}$; whence it is evident that the value of $x = 3$, and conſequently that $y = 5$, and $z = 4$; which are the numbers that were to be found.

PRO-

PROBLEM VIII.

To find the leaſt integer, poſſible, which being divided by 28, *ſhall leave a remainder of* 19; *but, being divided by* 19, *the remainder ſhall be* 15; *and, being divided by* 15, *the remainder ſhall be* 11.

Firſt, to find the leaſt whole number that can anſwer the two firſt conditions, let the quotient by 28, the firſt of the given diviſors, be denoted by x, or, which is the ſame, let the ſaid number be expreſſed by $28x + 19$; then this number, when 15 is ſubtracted from it, being diviſible by 19, it is manifeſt that $\frac{28x + 4}{19}$, or its equal $x + \frac{9x + 4}{19}$ muſt be an integer; from whence the leaſt value of x will be found $= 8$: and conſequently $28x + 19 = 243$; which is the leaſt whole number that can, poſſibly, ſatisfy the two firſt conditions. This being found, let the leaſt number that is exactly diviſible by both the ſaid diviſors 28 and 19, be now aſſumed; which, becauſe 28 and 19, are prime to each other, will be equal to 28×19, or 532: then, ſince the number required, by the nature of the problem, muſt be ſome multiple of 532, increaſed by 243, it is plain that the ſaid number may be repreſented by $532x + 243$; from which, if 11 be ſubtracted, and the remainder be divided by 15, the quotient $\left(\frac{532x + 232}{15} = 35x + 15 + \frac{7x + 7}{15}\right)$ will be a whole number by the queſtion, and conſequently $\frac{7x + 7}{15}$ a whole number alſo; from whence the leaſt value of x will be found $=$ 14, and conſequently that of $532x + 243 = 7691$; which is the number that was to be found. In the ſame manner the leaſt number, poſſible, may be found, which being ſucceſſively divided by four, or more given diviſors, ſhall leave given remainders.

PRO-

PROBLEM IX.

Suppoſing $87x + 256y = 15410$; *to determine the leaſt value of* x, *and the greateſt of* y, *in whole poſitive numbers.*

By tranſpoſition and diviſion we have

$y = \frac{15410-87x}{256} = 60 - \frac{87x-50}{256}$: where the fraction being the ſame with that in *Examp.* 1. to the premiſed *lemma*, the required value of x will be given from thence $= 30$; from whence that of y will likewiſe be known. But I ſhall in this place, ſhew the manner of deducing theſe values, independent of all previous conſiderations, by a method on which the demonſtration of the *lemma* itſelf depends.

In order to this, it is evident, as the quantity $87x - b$ (ſuppoſing $b = 50$) is diviſible by 256, that its double $174x - 2b$ muſt be likewiſe diviſible by 256. But $256x$ is, plainly, diviſible by 256; and, if from *this*, the quantity in the preceding line be ſubtracted, the remainder, $82x + 2b$ will be likewiſe diviſible by the ſame number; ſince *whatſoever number meaſures the whole, and one part of another, muſt do the like by the remaining part*: for which reaſon, if the quantity laſt found be ſubtracted from the firſt, the remainder $5x - 3b$ will alſo be diviſible by 256: and, if this new remainder multiplied by 16, be ſubtracted from the preceding one (in order to farther diminiſh the coefficient of x), the difference $2x + 50b$ muſt be, ſtill, diviſible by the ſame number. In like manner, the double of the laſt line, or remainder, being ſubtracted from the preceding one, we have $x - 103b$, a quantity, *ſtill*, diviſible by 256: but $\frac{103b}{256} = 20 + \frac{30}{256}$; therefore $x - 30$ muſt be diviſible by 256; and conſequently x be either equal to 30, or to 30 increaſed by ſome multiple of 256; but 30, being the leaſt value, is *that* required.

It may not be amiſs to add here another Example, to illuſtrate the way of proceeding by this laſt method: wherein let us ſuppoſe the quantity given to be $\frac{987x+651}{1235}$.

Then, making $b = 651$, the whole proceſs will ſtand as follows.

From - - - - -	$1235x$
ſub. - - - - -	$987x + b$
1. rem. - - - -	$248x - b$
1. rem. × 3 - -	$744x - 3b$
2. rem. - - - -	$243x + 4b$
3. rem. - - - -	$5x - 5b$
3. rem. × 48 -	$240x - 240b$
4. rem. - - - -	$3x + 244b$
5. rem. - - - -	$2x - 249b$
6. rem. - - - -	$x + 493b$;

where, x being without a coefficient, let $493b$ or its equal 320943 be now divided by 1235, the common meaſure to all thoſe quantities, and the remainder will be found 1078; therefore $x + 1078$ is likewiſe diviſible by 1235; and conſequently the leaſt value of x $(= 1235 - 1078) = 157$. The manner of working, according to this method, may be a little varied; it being to the ſame effect, whether the laſt remainder, or a *multiple of it*, be ſubtracted from the preceding one, or the preceding one, from ſome greater *multiple* of the laſt. Thus, in the example before us, the quantity $248x - b$, in the third line, might have been multiplied by 4, and the preceding one ſubtracted from the product; which would have given $5x - 5b$ (as in the ſixth line) by one ſtep leſs.—If the manner of proceeding in theſe two examples be compared with the proceſs for finding the ſame values, according to the *lemma*, the grounds of *this* will appear obvious.

PROBLEM X.

Suppoſing e, f, and g to denote given integers; to determine the value of x, ſuch that the quantities $\frac{x-e}{28}$, $\frac{x-f}{19}$, *and* $\frac{x-g}{15}$, *may all of them be integers.*

By making $\frac{x - e}{28} = y$, we have $x = 28y + e$; which value being ſubſtituted in our ſecond expreſſion, it becomes $\frac{28y + e - f}{19}$; which, as well as y, is to be a whole number: but $\frac{28y + e - f}{19}$, by making $b = e - f$, will be $= y + \frac{9y + b}{19}$; and therefore $19y$ and $18y + 2b$ being both diviſible by 19, their difference $y - 2b$ muſt be alſo diviſible by the ſame number; whence it is evident, that one value of y is $2b$; and that $2b + 19z$ (ſuppoſing z a whole number) will be a general value of y; and conſequently that x $(= 28y + e) = 532z + 56b + e$ is a general value of x, anſwering the two firſt conditions. Let this, therefore, be ſubſtituted in the remaining expreſſion $\frac{x - g}{15}$; which, by that means, becomes $\frac{532z + 56b + e - g}{15} = 35z + 3b + \frac{7z + \beta}{15}$ (ſuppoſing $\beta = 11b + e - g = 12e - 11f - g$.) Here $15z$ and $14z + 2\beta$ being both diviſible by 15, their difference $z - 2\beta$ muſt likewiſe be diviſible by the ſame number; and therefore one value of z will be 2β, and the general value of $z = 2\beta + 15w$: from whence the general value of x $(= 532z + 56b + e)$ is given $= 7980w + 1064\beta + 56b + e$; which, by reſtoring the values of b, and β, becomes $7980w + 12825e - 11760f - 1064g$.

Now, to have all the terms affirmative, and their coefficients the leaſt poſſible, let w be taken $= -e + 2f + g$; whence there reſults $4845e + 4200f + 6916g$, for a new value of x: from which, by expounding e, f, and g, by their given values, and dividing the whole by 7980, the leaſt value of x, which is the remainder of the diviſion, will be known.

PRO-

PROBLEM XI.

If $5x + 7y + 11z = 224$; *it is required to find all the poſſible values of* x, y, *and* z, *in whole numbers.*

In this, and other queſtions of the ſame kind, where you have three, or more indeterminate quantities and only one equation, it will be proper, firſt of all, to find the limits of thoſe quantities. Thus, in the preſent caſe, becauſe x is $= \frac{224 - 7y - 11x}{5}$, and becauſe the leaſt values of y and z cannot (by the queſtion) be leſs than unity, it is plain that x cannot be greater than $\frac{224 - 7 - 11}{5}$, or 41: and, in the ſame manner it will appear, that y cannot be greater than 29, nor z greater than 19; which therefore are the required limits in this caſe. Moreover, ſince x is $= \frac{224 - 7y - 11z}{5} = 45 - y - 2z - \frac{1 + 2y + z}{5} =$ a whole number, it is manifeſt that $\frac{2y + z + 1}{5}$ muſt alſo be a whole number: let $z + 1$ be therefore conſidered as a known quantity, and let the ſame be repreſented by b, and then the laſt expreſſion will become $\frac{2y + b}{5}$; from which, by proceeding as above, we ſhall get $y = 2b = 2z + 2$; whence the correſponding value of x comes out $= 42 - 5z$.

Let z be now taken $= 1$, then will $x = 37$ and $y = 4$; from the former of which values, let the coefficient of y be, continually, ſubtracted, and to the latter, let that of x be continually added, and we ſhall thence have 37, 30, 23, 16, 9, and 2, for the ſucceſſive values of x; and 4, 9, 14, 19, 24, and 29, for the correſponding values of y: which are all the poſſible anſwers when $z = 1$.

Let z be, now, taken $= 2$, then $x = 32$, and $y = 6$; let the former of theſe values be increaſed or decreaſed by the multiples of 7, and the latter by thoſe of 5, as far as poſſible, till they become negative; ſo ſhall we have 39, 32, 25, 18, 11, and 4, for the ſucceſſive values of x, in this caſe, and 1, 6, 11, 16, 21, and 26, for the reſpective values of y: which are all the anſwers when $z = 2$.

Again, let z be taken $= 3$; then, by proceeding as above, the correſponding values of x, and y will be found equal to 34, 27, 20, 13, 6; and 3, 8, 13, 18, 23, reſpectively. And ſo of the reſt: whence we have the following anſwers, being 60 in number.

z	y	x
1	4. 9. 14. 19. 24. 29.	37. 30. 23. 16. 9. 2.
2	1. 6. 11. 16. 21. 26.	39. 32. 25. 18. 11. 4.
3	3. 8. 13. 18. 23.	34. 27. 20. 13. 6.
4	5. 10. 15. 20. 25.	29. 22. 15. 8. 1.
5	2. 7. 12. 17. 22.	31. 24. 17. 10. 3.
6	4. 9. 14. 19.	26. 19. 12. 5.
7	1. 6. 11. 16.	28. 21. 14. 7.
8	3. 8. 13. 18.	23. 16. 9. 2.
9	5. 10. 15.	18. 11. 4.
10	2. 7. 12.	20. 13. 6.
11	4. 9. 14.	15. 8. 1.
12	1. 6. 11.	17. 10. 3.
13	3. 8.	12. 5.
14	5. 10.	14. 7.
15	2. 7.	9. 2.
16	4.	4.
17	1.	6.
18	3.	1.

PROBLEM XII.

If $17x + 19y + 21z = 400$; *it is propoſed to find all the poſſible values of* x, y, *and* z, *in whole poſitive numbers.*

When the coefficients of the indeterminate quantities x, y, and z, are nearly equal, as in this equation, it will be convenient to ſubſtitute for the ſum of thoſe quantities. Thus, let $x + y + z$ be put $= m$; then by ſubtracting 17 times this laſt equation from the preceding one, we ſhall have $2y + 4z = 400 - 17m$; and by ſubtracting the given equation from 21 times the aſſumed one $x + y + z = m$, there will remain $4x + 2y = 21m - 400$. Therefore, ſince y and z can have no values leſs than unity, it is plain, from the firſt of theſe two equations, that $400 - 17m$ cannot be leſs than 6, and therefore m not greater than $\frac{400 - 6}{17}$, or 23: alſo, becauſe by the ſecond of the two laſt equations, $21m - 400$ cannot be leſs than 6, it is obvious that m cannot be leſs than $\frac{400 + 6}{21}$, or 19: therefore 19 and 23 are the limits of m, in this caſe. Theſe being determined, let $4x$ be tranſpoſed in the laſt equation, and the whole be divided by 2, and we ſhall have $y = 10m - 200 - 2x + \frac{m}{2}$: which being a whole number, by the queſtion, $\frac{m}{2}$ muſt likewiſe be a whole number, and conſequently m an even number; which, as the limits of m are 19 and 23, can only be 20, or 22: let, therefore, m be firſt taken $= 20$, then y will become $= 10 - 2x$ and z $(m - x - y) = 10 + x$; wherein x being taken equal to 1, 2, 3, and 4, ſucceſſively, we ſhall have y equal to 8, 6, 4, 2, and z equal to 11, 12, 13, 14, reſpectively, which are four of the anſwers required. Again, let m be taken $= 22$; then will $y = 31 - 2x$, and $z = x - 9$; wherein let x be interpreted by 10, 11, 12, 13, 14, and 15, ſucceſſively, whence y will come out 11, 9, 7, 5, 3, and 1;

and x equal to 1, 2, 3, 4, 5, and 6, reſpectively. Therefore we have the ten following anſwers; which are all the queſtion admits of.

$x =$	1	2	3	4	10	11	12	13	14	15
$y =$	8	6	4	2	11	9	7	5	3	1
$z =$	11	12	13	14	1	2	3	4	5	6

PROBLEM XIII.

Suppoſing $7x + 9y + 23z = 9999$; *it is required to determine the number of all the anſwers, in poſitive integers.*

In caſes like *this*, where the anſwers are very many, and the number of them *only* is required, the following method may be uſed.

In the general equation $ax + by + cz = k$ (where a and b are ſuppoſed prime to each other) let z be aſſumed $= 0$; and find the greateſt value of x, and the leaſt of y, in the equation $ax + by = k$, thence ariſing; denoting them by g and l: find, moreover, the leaſt poſitive value of n (in whole numbers) from the equation $am + bn = c$, together with the correſponding value of m, whether poſitive or negative; then, ſuppoſing q to repreſent an integer, the general value of x may be expreſſed by $g - bq - mz$, and that of y by $l + aq - nz$; as will appear by ſubſtituting in the general expreſſion $ax + by + cz$, which thereby becomes $ag - abq - amz + bl + abq - bnz + cz = k$ (as it ought to be), becauſe $ag + bl = k$, and all the reſt of the terms deſtroy one another. And it may be obſerved farther, by the bye, and is evident from hence, that any two correſponding values of m and n, determined from the equation $am + bn = c$, will equally fulfil the conditions of the general equation; but the leaſt are to be uſed, as being the moſt commodious.—As to the limits of z and q, theſe are eaſily determined; the former from the original equation, and the latter from the general value of x; by which it appears that q cannot exceed $\frac{g - mz}{b}$; wherein the greateſt, or the leaſt value of z is to be uſed, according as the ſecond term, after ſubſtitution for m, is *poſitive* or *negative*. But, beſides *this*, there is another limit, or particular value of q to be determined, which is of great uſe in finding the number of anſwers.

It is evident from the given equations, that the values of x will begin to be negative, when z is ſo increaſed as to exceed $\frac{g - bq}{m}$; and that thoſe of y will, in like manner, become negative, when z is taken greater than $\frac{l + aq}{n}$: therefore, as long as $\frac{g - bq}{m}$ continues greater than $\frac{l + aq}{n}$ (ſuppoſing the value of q to be varied) ſo long will x admit of a greater aſſumption for z than y will admit of, without producing negative values; and *vice verſâ*. By making, therefore, theſe two expreſſions equal to each other, the value of q will be given $\left(= \frac{ng - ml}{am + nb}\right) = \frac{ng - ml}{c}$; expreſſing the circumſtance wherein both the values of x and y, by increaſing z, become negative together. But this holds, only, when m is a poſitive quantity; for, in the other caſe, the laſt term ($- mz$) in the general value of x being poſitive, the particular values do not become negative by increaſing, but by diminiſhing the value of z; it being evident, that no ſuch can reſult from any aſſumption for z, but when q is greater than $\frac{g}{b}$.

To apply theſe obſervations to the equation, $7x + 9y + 23z = 9999$, propoſed, we ſhall, in the firſt place, by taking $z = 0$, have $x = 1428 - y - \frac{2y - 3}{7}$: whence the leaſt value of y is given $= 5$; and the greateſt of $x = 1422$. Again, from the equation $am + bn = c$, or $7m + 9n = 23$, we have $m = 3 - n - \frac{2n - 2}{7}$; in which the leaſt, poſitive, value of n is given $= 1$; and the correſponding value of $m = 2$; and ſo the general values of x and y do here become $1422 -$

$9q - 2z$, and $5 + 7q - z$, reſpectively. From the former of which the greater limit of q is given $= \frac{1422 - 2}{9}$, or $157\frac{7}{9}$; and from $\frac{ng - ml}{c}$, expreſſing the leſſer limit, we have 61, for the value of q, when the leaſt value of x becomes equal to that of y. Theſe limits being aſſigned, let q be now interpreted by 0, 1, 2, 3, 4, 5, &c. ſucceſſively, up to 61, incluſive: whence the number of anſwers, or variations of y correſponding to every interpretation, will be found as in the margin. From whence it appears that the arithmetical progreſſion $4 + 11 + 18 + 25 + 32$, &c. continued to 62 terms, will truly expreſs the number of all the anſwers when q is leſs than 62: which number is therefore given $= \overline{4 + 61 \times 7 + 4} \times 31 = 13485$.

q	$y =$	N. Anſ.
0	$5 - z$	4
1	$12 - z$	11
2	$19 - z$	18
3	$26 - z$	25
4	$33 - z$	32
&c.	&c.	&c.

In all which anſwers it is evident, that x, as well as y, will be poſitive (as it ought to be): becauſe it has been proved that the leaſt value of x, till q becomes $\left(= \frac{ng - ml}{c}\right) = 61\frac{2}{3}$, will be greater than that of y; which is poſitive, ſo far. But now, to find the anſwers when q is upwards of 61, we muſt have recourſe to the general value of x; which, in theſe caſes, by the different interpretations of z, becomes negative before that of y.

q	$x =$	$z \sqsupset$	N. Anſ.
157	$9 - 2z$	$4\frac{1}{2}$	4
156	$18 - 2z$	9	8
155	$27 - 2z$	$13\frac{1}{2}$	13
154	$36 - 2z$	18	17
153	$45 - 2z$	$22\frac{1}{2}$	22
&c.	&c.	&c.	&c.

Here, by beginning with the greateſt limit, and writing 157, 156, 155, 154, &c. ſucceſſively, in the room of q, it will appear, that the number of anſwers will be truly expreſſed by the ſeries $4 + 8 + 13 + 17 + 22$ &c. continued to $157 - 61$ terms: which terms being united in pairs (becauſe, in every two terms, the ſame fraction in the limit of z occurs) the ſeries 12 + 30 + 48 + &c. thence ariſing, will be a true arithmetical progreſſion; whereof the common difference being 18, and the number of terms $= \frac{157-61}{2} = 48$, the ſum will therefore be given $= 20880$: to which adding 13485, the number of anſwers when q was leſs than 62, the aggregate 34365 will be the whole number of all the anſwers required.

PROBLEM XIV.

To determine how many different ways it is poſſible to pay 1000l. without uſing any other coin than crowns, guineas, and moidores.

By the conditions of the problem we have $5x + 21y + 27z = 20000$; where taking $z = 0$, x is found $= 4000 - 4y - \frac{y}{5}$, and from thence the leaſt value of $y = 0$ (0 being to be included, *here*, by the queſtion): whence the greateſt value of x is given $= 4000$. Moreover, from the equation $5m + 21n = 27$, we have $m = 5 - 4n - \frac{n-2}{5}$; from which $n = 2$, and $m = -3$: ſo that the general values of x and y, given in the preceding problem, will here become $4000 - 21q + 3z$, and $5q - 2z$. Moreover, from the given equation, the greateſt limit of z appears to be $= \frac{20000}{27} = 740$; whence we alſo have $\frac{g-mz}{b} = \frac{4000 + 3 \times 740}{21} = 296 =$ the greateſt limit of q; and $\frac{g}{b} = \frac{4000}{21} = 190$, expreſſing the leſſer limit of q, when the value of x, anſwering to ſome interpretations of z, will become negative, while thoſe of y ſtill continue affirmative. To find the number of all theſe affirmative values, up to the greateſt limit of q, let 0, 1, 2, 3, 4, 5, &c. be now wrote in the room of q (as in the margin). Whence it is evident that the ſaid number is compoſed of the ſeries 1 + 3 + 6 + 8 + 11 + 13 *&c.* continued to 297 terms; which terms (ſetting aſide the firſt) being united in pairs, we ſhall have the arithmetical progreſſion 9 + 19 + 29 *&c.* where the number of terms to be taken being 148, and common difference 10, the laſt term will therefore be 1479, and the ſum of the whole progreſſion 110112: to which adding (1) the term omitted, we have 110113, for the number of all the anſwers, including thoſe wherein the value of x is negative; which laſt muſt therefore be found and deducted.

q	$y =$	Quot.	N. Anſ.
0	$0 - 2z$	0	1
1	$5 - 2z$	$2\frac{1}{2}$	3
2	$10 - 2z$	5	6
3	$15 - 2z$	$7\frac{1}{2}$	8
4	$20 - 2z$	10	11
5	$25 - 2z$	$12\frac{1}{2}$	13
&c.	*&c.*	*&c.*	*&c.*

In order to this we have already found, that theſe negative values do not begin to have place till q is greater than 190: let, therefore, 191, 192, 193, *&c.* be ſubſtituted, ſucceſſively, for q; from whence it will appear that the number of all the ſaid negative values is truly exhibited by the arithmetical progreſſion 4 + 11 + 18 + 25 *&c.* continued to 296 — 190 terms; whereof the ſum is 39379; which ſubtracted from 110113, found above, leaves 70734, for the number of anſwers required.

q	$x.$	Quot.	N. Anſ.
191	$3z - 11$	$3\frac{2}{3}$	4
192	$3z - 32$	$10\frac{2}{3}$	11
193	$3z - 53$	$17\frac{2}{3}$	18
194	$3z - 74$	$24\frac{2}{3}$	25
&c.	*&c.*	*&c.*	*&c.*

After the manner of theſe two examples (which illuſtrate the two different caſes of the general ſolution, given in the preceding problem) the number of anſwers may be found in other equations, wherein there are three indeterminate quantities. But, in ſumming up the numbers ariſing from the different interpretations of q, due regard muſt be had to the fractions exhibited in the third column expreſſing the limits of z; becauſe, to have a regular progreſſion, the terms of the ſeries in the fourth column, exhibiting the number of anſwers, muſt be united by twos, threes, or fours, &c. according as one and the ſame fraction occurs every ſecond, third, or fourth, &c. term (the odd terms, when there happen any over, being always to be ſet aſide, at the beginning of the ſeries). And it may be obſerved farther, that, to determine the ſum of the progreſſion thus ariſing, it will be ſufficient to find the firſt term only, by an actual addition; ſince, not only the number of terms, but the common difference alſo, will be known; being always equal to the common difference of the limits of z (or of the quotients in the ſaid third column) multiplied by the ſquare of the number of terms united into one; whereof the reaſon is evident. But all *this* relates to the caſes wherein the coefficients of the indeterminate quantities, in the given equations, are (two of them at leaſt) prime to each other: I ſhall add one example more, to ſhew the way of proceeding when thoſe coefficients admit of a common meaſure.

PROBLEM XV.

Suppoſing $12x + 15y + 20z = 100001$; *it is required to find the number of all the anſwers in poſitive integers.*

It is evident, by tranſpoſing $20z$ and dividing by (3) the greateſt common meaſure of x and y, that $4x + 5y$, and conſequently it's equal $33333 - 6z - \frac{2z - 2}{3}$, muſt be an integer, and therefore $2z - 2$ diviſible by 3: but $3z$ is diviſible by 3, and ſo the difference of theſe two, which is $z + 2$, muſt be likewiſe diviſible by the ſame number, and conſequently $z = 1 +$ ſome multiple of 3. Make, therefore, $1 + 3u = z$ (u being an integer); then the given equation, by ſubſtituting this value, will become $12x + 15y + 60u + 20 = 100001$; which, by diviſion, &c. is reduced to $4x + 5y + 20u = 33327$: wherein the coefficients of x and y are now prime to each other, and we are to find the number of all the variations, anſwering to the different interpretations of u, from 0 to the greateſt limit, incluſive.

By proceeding, therefore, as in the aforegoing caſes, we have $x = 8331 - y - \frac{y - 3}{4}$; whence the leaſt value of y is given $= 3$, and the greateſt of $x = 8328$. Moreover, from the equation $4m + 5n = 20$, we have $m = 5 - n - \frac{n}{4}$; whence $n = 0$, and $m = 5$. Therefore the general values of x and y (*given in Problem* 13) do here become $8328 - 5q - 5u$, and $3 + 4q$; from the former of which the greateſt limit of q is given $= \frac{8328}{5} = 1665$. Now, ſince the value of y will here continue poſitive, in all ſubſtitutions for q and u (as no negative quantity enters therein); the whole number of anſwers will be determined by the values of x alone.

In order to this, let q be ſucceſſively expounded by 1665, 1664, 1663, *&c.* and it will thence appear that the ſaid number will be truly defined by 1666 terms of the arithmetical progreſſion 1 + 2 + 3 + 4 + 5 *&c.* whereof the ſum is found to be 1388611.

q	x	Quot.	N.Anſ.
1665	$3 - 5u$	$0\frac{3}{5}$	1
1664	$8 - 5u$	$1\frac{3}{5}$	2
1663	$13 - 5u$	$2\frac{3}{5}$	3
&c.	*&c.*	*&c.*	*&c.*

When there are four indeterminate quantities in the given equation, the number of all the anſwers may be determined by the ſame methods: for, any one of thoſe quantities may be interpreted by all the integers, ſucceſſively, up to its greateſt limit (which is eaſily determined); and the number of anſwers, correſponding to each of theſe interpretations may be found, as above; the aggregate of all which will conſequently be the whole number of anſwers required: which ſum, or aggregate, may, in many caſes, be derived by the methods given in Section 14, for ſumming of ſeries's by means of a known relation of their terms. But this being a matter of more ſpeculation than real uſe, I ſhall now paſs on to other ſubjects.

SECTION XIV.

The Inveſtigation of the ſums of powers of Numbers in arithmetical progreſſion.

BESIDES the two ſorts of progreſſions treated of in Section 10, there are infinite varities of other kinds; but the moſt uſeful, and the beſt known, are thoſe conſiſting of the powers of numbers in arithmetical progreſſion; ſuch as $1^2 + 2^2 + 3^2 + 4^2 \ldots\ldots n^2$, and $1^3 + 2^3 + 3^3 + 4^3 \ldots\ldots n^3$, *&c.* where *n* denotes the number of terms to which each progreſſion is to be continued. In order to inveſtigate the ſum of any ſuch progreſſion, which is the deſign of this ſection, it will be requiſite, firſt of all, to premiſe the following

LEMMA.

If any expreſſion, or ſeries, as

$$\left.\begin{array}{l} An + Bn^2 + Cn^3 + Dn^4 \;\&c. \\ -an - bn^2 - cn^3 - dn^4 \;\&c. \end{array}\right\}$$

, involving the powers of an indeterminate quantity *n*, be univerſally equal to nothing, whatſoever be the value of *n*; then, I ſay, the ſum of the coefficients $A - a$, $B - b$, $C - c$, *&c.* of each rank of homologous terms, or of the ſame powers of *n*, will alſo be equal to nothing.

For, in the firſt place, let the whole equation

$$\left.\begin{array}{l} An + Bn^2 + Cn^3 \;\&c. \\ -an - bn^2 - cn^3 \;\&c. \end{array}\right\} = 0,$$

be divided by *n*, and we ſhall have

$$\left\{\begin{array}{l} A + Bn + Cn^2 \;\&c. \\ -a - bn - cn^2 \;\&c. \end{array}\right\} = 0;$$

and this being univerſally ſo, be the value of *n* what it will, let, therefore, *n* be taken $= 0$, and it will become

$$\left\{\begin{array}{c} A \\ -a \end{array}\right\} = 0;$$

which being rejected, as ſuch, out of the laſt equation, we ſhall next have

$$\left.\begin{array}{l} +Bn + Cn^2 + Dn^3 \;\&c. \\ -bn - cn^2 - dn^3 \;\&c. \end{array}\right\} = 0;$$

whence, dividing

again by n, and proceeding in the very ſame manner, $B - b$ is alſo proved to be $= 0$; and from thence $C - c$, $D - d$, &c. &c. Q. E. D.

Now, to apply what is here demonſtrated to the purpoſe above ſpecified, it will be proper to obſerve, firſt, that, as the value of any progreſſion $(1^2 + 2^2 + 3^2 + 4^2 \ldots\ldots n^2)$ varies according as (n) the number of its terms varies, it muſt (if it can be expreſſed in a general manner) be explicable by n and its powers with determinate coefficients; ſecondly, it is obvious that thoſe powers, in the caſes above propoſed, muſt be rational, or ſuch whoſe indices are whole poſitive numbers; becauſe the progreſſion, being an aggregate of whole numbers, cannot admit of ſurd quantities; laſtly, it will appear that the greateſt of the ſaid indices cannot exceed the common index of the progreſſion by more than unity; for, otherwiſe, when n is taken indefinitely great, the higheſt power of n would be indefinitely greater than all the reſt of the terms put together.

Thus, the higheſt power of n, in an expreſſion univerſally exhibiting the value of $1^2 + 2^2 + 3^2 \ldots\ldots n^2$, cannot be greater than n^3; for $1^2 + 2^2 + 3^2 \ldots\ldots n^2$ is manifeſtly leſs than n^3 (or $n^2 + n^2 + n^2 +$ &c. continued to n terms); but n^4, when n is indefinitely great, is indefinitely greater than n^3, or any other inferior power of n, and therefore cannot enter into the equation. This being premiſed, the method of inveſtigation may be as follows.

Caſe 1°. *To find the ſum of the progreſſion* $1 + 2 + 3 + 4 \ldots\ldots n$.

Let $An^2 + Bn$ be aſſumed according to the foregoing obſervations, as an univerſal expreſſion for the value of $1 + 2 + 3 + 4 \ldots\ldots n$; where A and B repreſent unknown, but determinate quantities. Therefore, ſince the equation is ſuppoſed to hold univerſally, whatſoever is the number of terms, it is evident, that, if the number of terms be increaſed by unity, or, which is the ſame thing, if $n+1$ be wrote therein, inſtead of n, the equality will ſtill ſubſiſt, and we ſhall have $A\times\overline{n+1}|^{2}+B\times\overline{n+1}=1+2+3+4\ldots\ldots n+\overline{n+1}$. From which the firſt equation being ſubtracted, there remains $A\times\overline{n+1}|^{2}-An^{2}+B\times\overline{n+1}-Bn=n+1$: this contracted will be $2An+A+B=n+1$; whence we have $\overline{2A-1}\times n+A+B-1=0$: wherefore, by taking $2A-1=0$, and $A+B-1=0$ (*according to the lemma*) we have $A=\frac{1}{2}$, and $B=\frac{1}{2}$; and conſequently $1+2+3+4\ldots\ldots n\ (=An^{2}+Bn)=$ $\frac{n^{2}}{2}+\frac{n}{2}$, or $\frac{n\times\overline{n+1}}{2}$[*].

Caſe 2°. *To find the ſum of the progreſſion* $1^{2}+2^{2}+3^{2}\ldots\ldots n^{2}$, or $1+4+9+16\ldots\ldots n^{2}$.

Let $An^{3}+Bn^{2}+Cn$, according to the aforeſaid obſervations, be aſſumed $=1^{2}+2^{2}+3^{2}+4^{2}\ldots\ldots n^{2}$: then, by reaſoning as in the preceding caſe, we ſhall have $A\times\overline{n+1}|^{3}+B\times\overline{n+1}|^{2}+C\times\overline{n+1}=1^{2}+2^{2}+3^{2}+4^{2}\ldots\ldots n^{2}+\overline{n+1}|^{2}$; that is, by involving $n+1$ to its ſeveral powers, $An^{3}+3An^{2}+3An+A+Bn^{2}+2Bn+B+Cn+C=1^{2}+2^{2}+3^{2}+4^{2}\ldots\ldots n^{2}+\overline{n+1}|^{2}$: from which, ſubtracting the former equation, we get $3An^2 + 3An + A + 2Bn + B + C$ $(= \overline{n+1}|^2) = n^2 + 2n + 1$; and conſequently $\overline{3A - 1} \times n^2 + \overline{3A + 2B - 2} \times n + A + B + C - 1 = 0$; whence (*by the lemma*) $3A - 1 = 0$, $3A + 2B - 2 = 0$, and $A + B + C - 1 = 0$; therefore $A = \frac{1}{3}$, $B = \frac{2 - 3A}{3} = \frac{1}{2}$, $C = 1 - A - B = \frac{1}{6}$; and conſequently $1 + 4 + 9 + 16 \ldots\ldots n^2 = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}$, or $\frac{n \cdot \overline{n+1} \cdot \overline{2n+1}}{6}$.

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[*] In this inveſtigation it is taken for granted, that the ſum of the progreſſion is capable of being exhibited by means of the powers of n, with proper coefficients; which aſſumption is verified by the proceſs itſelf; for it is evident from thence, that the quantities $An^{2}+Bn$, and $1+2+3+4\ldots n$, under the values of A and B *there* determined, are *always* increaſed equally, by taking the value of n greater by an unit: if, therefore, they are equal to each other, when n is $=0$ (as they actually are) they muſt alſo be equal when n is 1; and ſo likewiſe, when n is 2, *&c. &c.* And the ſame reaſoning holds in all the following caſes.

Caſe 3°. *To determine the ſum of the progreſſion* $1^3 + 2^3 + 3^3 + 4^3 \ldots\ldots n^3$, *or* $1 + 8 + 27 + 64 \ldots\ldots n^3$.

By putting $An^4 + Bn^3 + Cn^2 + Dn = 1 + 8 + 27 + 64 \ldots\ldots n^3$, and proceeding as above, we ſhall have $4An^3 + 6An^2 + 4An + A + 3Bn^2 + 3Bn + B + 2Cn + C + D \; (= \overline{n+1}|^3) = n^3 + 3n^2 + 3n + 1$; and therefore $\overline{4A - 1} \times n^3 + \overline{6A + 3B - 3} \times n^2 + \overline{4A + 3B + 2C - 3} \times n + A + B + C + D - 1 = 0$: hence $A = \frac{1}{4}$, $B \left(= \frac{3 - 6A}{3}\right) = \frac{1}{2}$, $C \left(= \frac{3 - 4A - 3B}{2}\right) = \frac{1}{4}$, $D \; (= 1 - A - B - C) = 0$; and therefore $1^3 + 2^3 + 3^3 + 4^3 \ldots\ldots n^3 = \frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4}$, or $\frac{n^2 \times \overline{n+1}|^2}{4}$. In the very ſame manner it will be found, that

$$1^4 + 2^4 + 3^4 \ldots\ldots n^4 = \frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30},$$

$$1^5 + 2^5 + 3^5 \ldots\ldots n^5 = \frac{n^6}{6} + \frac{n^5}{2} + \frac{5n^4}{12} - \frac{n^2}{12}.$$

$$1^6 + 2^6 + 3^6 \ldots\ldots n^6 = \frac{n^7}{7} + \frac{n^6}{2} + \frac{n^5}{2} - \frac{n^3}{6} + \frac{n}{42}.$$

&c. &c.

In order to exemplify what has been thus far delivered, let it, in the firſt place, be required to find the ſum of the ſeries of ſquares $1 + 4 + 9 + 16$ &c. continued to 10 terms: then, by ſubſtituting 10 for n, in the general expreſſion $\frac{n\cdot\overline{n+1}\cdot\overline{2n+1}}{6}$ (or $\frac{n^3}{3}+\frac{n^2}{2}+\frac{n}{6}$), found by caſe 2°, there will come out 385, for the required ſum of the progreſſion: which, the number of terms being here ſmall, may be eaſily confirmed, by actually adding the 10 terms together. Secondly, let it be required to find the number of cannon-ſhot in a ſquare pile whoſe ſide is 50; then, by writing 50 for n in the ſame expreſſion, $\frac{n\cdot\overline{n+1}\cdot\overline{2n+1}}{6}$, we ſhall have $\left(\frac{50\times 51\times 101}{6}\right)$ 42925, expreſſing the number of ſhot in ſuch a pile. Laſtly, ſuppoſe a pyramid compoſed of 100 ſtones of a cubical figure; whereof the length of the ſide of the higheſt is one inch; of the ſecond two inches; of the third three inches, &c. Here, by writing 100 inſtead of n, in the third general expreſſion, we have 25502500, for the number of ſolid inches in ſuch a pyramid.

Hitherto regard has been had to ſuch progreſſions as have unity for their firſt term, and likewiſe for the common difference; but the ſame equations, or theorems, with very little trouble, may be alſo extended to thoſe caſes where the firſt term, and the common difference, are any given numbers, provided the former of them be any multiple of the latter. Thus, ſuppoſe it were required to find the ſum of the progreſſion $6^2 + 8^2 + 10^2$ &c. (or $36 + 64 + 100$ &c.) continued to eight terms: then, by making (4), the ſquare of the common difference, a general multiplicator, the given expreſſion will be reduced to $4 \times \overline{3^2 + 4^2 + 5^2 \ldots\ldots 10^2}$: but the ſum of the progreſſion $1^2 + 2^2 + 3^2 + 4^2 \ldots\ldots 10^2$ is found, by the ſecond Theorem, to be 385; from which, if (5) the ſum of the two firſt terms (which the ſeries $3^2 + 4^2 + 5^2 \ldots\ldots 10^2$ wants) be taken away; the remainder will be 380; and this, multiplied by 4, gives 1520, for the true ſum of the propoſed progreſſion; and ſo of others.

But if the firſt term is not diviſible by the common difference, as in the progreſſion, $5^2 + 7^2 + 9^2$ &c. the ſpeculation is a little more difficult; nevertheleſs, the ſum of the ſeries, in any ſuch caſe, may be, ſtill, found, from the ſame Theorems.

Let the ſeries $\overline{m+e}|^2 + \overline{m+2e}|^2 + \overline{m+3e}|^2 \ldots\ldots \overline{m+ne}|^2$ be propoſed, where m and e denote any quantities whatever, and where n repreſents the number of terms. Then, by actually raiſing each root to its ſecond power, and placing the terms in order, the given expreſſion will ſtand thus

$$\left.\begin{matrix} m^2 + m^2 + m^2 \ldots\ldots m^2 \\ 2me + 4me + 6me \ldots\ldots 2nme \\ e^2 + 4e^2 + 9e^2 \ldots\ldots n^2e^2 \end{matrix}\right\}.$$

Now, it is evident that the ſum of the firſt rank, or ſeries, is $n \times m^2$: alſo the ſum of the ſecond, or $2me \times \overline{1+2+3+4 \ldots\ldots n}$ appears (*by caſe* 1) to be $2me \times \frac{n \times \overline{n+1}}{2}$; and that of the third, or $e^2 \times \overline{1+4+9+16 \ldots\ldots n^2}$ (*by caſe* 2) $= e^2 \times \frac{n . \overline{n+1} . \overline{2n+1}}{6}$: therefore the ſum of the whole progreſſion, $\overline{m+e}|^2 + \overline{m+2e}|^2 + \overline{m+3e}|^2 \ldots\ldots \overline{m+ne}|^2$ is $= n . m^2 + n . \overline{n+1} . me + \frac{n . \overline{n+1} . \overline{2n+1} . e^2}{6}$.

In like manner, if the ſeries propoſed be $\overline{m+e}|^3 + \overline{m+2e}|^3 + \overline{m+3e}|^3 \ldots\ldots \overline{m+ne}|^3$; then may it be reſolved

$$\text{into}\left\{\begin{matrix} \overline{1+1+1 \ldots\ldots 1} \times \overline{m^3} \\ \overline{1+2+3 \ldots\ldots n} \times \overline{3m^2e} \\ \overline{1+4+9 \ldots\ldots n^2} \times \overline{3me^2} \\ \overline{1+8+27 \ldots\ldots n^3} \times \overline{e^3} \end{matrix}\right\}$$

: whoſe ſum, by

the aforementioned Theorems, will appear to be $n \cdot m^3 + \frac{n \cdot \overline{n+1} \cdot 3m^2e}{2} + \frac{n \cdot \overline{n+1} \cdot \overline{2n+1} \cdot me^2}{2} + \frac{n^2 \cdot \overline{n+1}^2 \cdot e^3}{4}$. And, by following the ſame method, the ſums of other ſeries's may be determined, not only of powers, but likewiſe of rectangles, and ſolids, &c. provided that, their ſides, or factors, are in arithmetical progreſſion. Thus, for example, let there be propoſed the ſeries of rectangles $\overline{m+e} \cdot \overline{p+e} + \overline{m+2e} \cdot \overline{p+2e} + \overline{m+3e} \cdot \overline{p+3e} \ldots + \overline{m+ne} \cdot \overline{p+ne}$. Then, the factors being actually multiplied together, and the terms placed in order, the given ſeries will be reſolved into the three following ones:

$$mp + mp + mp + mp \ldots + mp$$
$$\overline{m+p} \cdot e + \overline{m+p} \cdot 2e + \overline{m+p} \cdot 3e + \overline{m+p} \cdot 4e \ldots + \overline{m+p} \cdot ne$$
$$e^2 + 4e^2 + 9e^2 + 16e^2 \ldots + n^2e^2.$$

Whereof the, reſpective, ſums (*by caſe* 1 *and* 2) are $mp \times n$, $\overline{m+p} \cdot e \times \frac{n \cdot \overline{n+1}}{2}$, and $e^2 \times \frac{n \cdot \overline{n+1} \cdot \overline{2n+1}}{6}$ and the aggregate of all theſe, or $n \times \overline{mp + \frac{n+1}{2} \cdot \overline{m+p} \cdot e + \frac{\overline{n+1} \cdot \overline{2n+1}}{6}} \cdot e^2$, is conſequently the true ſum of the ſeries of rectangles propoſed.

From this laſt general expreſſion, the number of cannon-ſhot in an oblong pile, whether whole or broken, will be known. For, ſuppoſing $e = 1$, our ſeries of rectangles becomes $\overline{m+1} \cdot \overline{p+1} + \overline{m+2} \cdot \overline{p+2} + \overline{m+3} \cdot \overline{p+3} \ldots\ldots + \overline{m+n} \cdot \overline{p+n}$; and the ſum thereof $= n \times \overline{mp + \frac{n+1}{2} \cdot \overline{m+p} + \frac{\overline{n+1} \cdot \overline{2n+1}}{6}} =$ the number ſought: where $m+1$ and $p+1$ repreſent the length and breadth of the uppermoſt rank, or tire; n being the number of ranks one above another.

But the expreſſion here brought out may be reduced to $\frac{n}{4} \times \overline{\overline{2m+n+1} . \overline{2p+n+1} + \frac{\overline{n+1} . \overline{n-1}}{3}}$; which is better adapted to practice, and which, expreſſed in words, gives the following rule.

To twice the length, and to twice the breadth of the uppermoſt rank, add the number of ranks leſs one, and multiply the two ſums together; alſo multiply the number of ranks leſs one, by that number more one, and add $\frac{1}{3}$ of this product to the former; then $\frac{1}{4}$ of the ſum multiplied by the number of ranks will be the anſwer.

As a rule of this ſort is of frequent uſe to perſons concerned in artillery, it may not be improper to add an example or two, by way of illuſtration.

1. Suppoſe a complete pile, conſiſting of 15 tires, or ranks, and ſuppoſe the number of ſhot in the uppermoſt (which in this caſe is a ſingle row) to be 32. Then the firſt product, mentioned in the rule will be $\overline{64+14} \times \overline{2+14} = 78 \times 16 = 1248$; and the ſecond $= 14 \times 16 = 224$; $\frac{1}{3}$ whereof is $74\frac{2}{3}$, and this, added to 1248, gives $1322\frac{2}{3}$; whereof $\frac{1}{4}$ part is $330\frac{2}{3}$; which, multiplied by 15, gives 4960, for the whole number of ſhot in ſuch a pile.

2. Let the pile be a broken one, ſuch that the length and breadth of the uppermoſt tire may be 25 and 16, and the number of tires 11.

Here, we have $\overline{50+10} \times \overline{32+10} = 60 \times 42 = 2520$, for the firſt product; and $12 \times 10 = 120$, for the ſecond: therefore $\frac{2560}{4} \times 11 = 640 \times 11 = 7040$, is the true anſwer.

Having exemplified the uſe of the Theorem, for finding the ſum of a ſeries of rectangles, I ſhall here ſubjoin one inſtance of *that* preceding *it*, for determining the ſum of a ſeries of cubes; wherein the value of the firſt 10 terms of the progreſſion $\overline{2+\sqrt{2}}|^3 + \overline{3+2\sqrt{2}}|^3 + \overline{4+3\sqrt{2}}|^3 + \overline{5+4\sqrt{2}}|^3$ &c. is required. Here,

e being $1 + \sqrt{2}$, m will be $= 1$; therefore, by writing 10, 1, and $1 + \sqrt{2}$ for n, m, and e, reſpectively, in the general expreſſion, it will become $10 + \frac{10 . 11 . 3 . \overline{1 + \sqrt{2}}}{2}$ $+ \frac{10 . 11 . 21 . \overline{1 + \sqrt{2}}|^2}{2} + \frac{100 . 121 . \overline{1 + \sqrt{2}}|^3}{4} =$ $24815 + 17600\sqrt{2}$, the value ſought.

If any one is deſirous to ſee this ſpeculation carried further, ſo as to extend to ſeries's of powers whoſe indices are fractions; ſuch as ſquare roots, cube roots, *&c.* I muſt beg leave to refer to my *Eſſays*, where it is treated in a general manner. Here I muſt deſire the reader to obſerve, once for all, that the Theorems above found will hold equally, in caſe of a deſcending ſeries, ſuch as $\overline{m - e}|^2 + \overline{m - 2e}|^2$ *&c.* or $\overline{m - e}|^3 + \overline{m - 2e}|^3$ *&c.* provided the ſigns of the ſecond, fourth, *&c.* terms be changed; as is evident from the inveſtigation.

Although the ſubject of this Section has, already, been pretty largely inſiſted on, yet it may not be improper to add a different method, whereby the ſame concluſions will, in many caſes, be more eaſily derived: in order to which it is neceſſary to premiſe the ſubſequent

LEMMA.

If $a + b + c + d + e +$ *&c.* be a ſeries, whereof the terms, a, b, c, d, &c. are ſo related to each other, that the ſum, or value thereof, can be univerſally expounded by an expreſſion of this form, *viz.* $An + B \times n \times \overline{n - 1} + C \times n \times \overline{n - 1} \times \overline{n - 2} + D \times n \times \overline{n - 1} \times \overline{n - 2} \times \overline{n - 3}$ *&c.* n being the number of terms to which the ſeries is to be continued, and A, B, C, D, *&c.* determinate coefficients; then, I ſay, the values of thoſe coefficients will be as hereunder ſpecified, *viz.*

$$A = a$$

$$B = \frac{-a + b}{2},$$

$$C = \frac{a - 2b + c}{2 \cdot 3},$$

$$D = \frac{-a + 3b - 3c + d}{2 \cdot 3 \cdot 4},$$

$$E = \frac{a - 4b + 6c - 4d + e}{2 \cdot 3 \cdot 4 \cdot 5},$$

&c. &c.

For, ſince the equation $A \times n + B \times n \times \overline{n-1} + C \times n \times \overline{n-1} \times \overline{n-2} + D \times n \times \overline{n-1} \times \overline{n-2} \times \overline{n-3}$ &c. $= a + b + c + d + e$, &c. is ſuppoſed to hold univerſally, let the number of terms be what it will, let n be expounded by 1, 2, 3, 4, &c. ſucceſſively, and the general equation will become

1°. $^{*}A = a,$
2°. $2A + 2B = a + b,$
3°. $3A + 6B + 6C = a + b + c,$
4°. $4A + 12B + 24C + 24D = a + b + c + d,$
5°. $5A + 20B + 60C + 120D + 120E = a + b + c + d + e,$
&c. &c.

Now, the double of the firſt of theſe equations being ſubtracted from the ſecond, its triple from the third, and its quadruple from the fourth, &c. we ſhall have

$$^{*}2B = b - a,$$
$$6B + 6C = -2a + b + c,$$
$$12B + 24C + 24D = -3a + b + c + d,$$
$$20B + 60C + 120D + 120E = -4a + b + c + d + e,$$

&c. &c.

Again, if the triple of the firſt of theſe be ſubtracted from the ſecond, and its ſextuple from the third, &c. we ſhall, next, have

$$^{*}6C = a - 2b + c,$$
$$24C + 24D = 3a - 5b + c + d,$$
$$60C + 120D + 120E = 6a - 9b + c + d + e.$$

Moreover by taking the quadruple of the firſt of theſe from the ſecond, &c. we get

$$^{*}24D = -a + 3b - 3c + d, \text{ and}$$
$$120D + 120E = -4a + 11b - 9c + d + e;$$

from the latter of which ſubtract the quintuple of the former, and there will remain

$$*120E = a - 4b + 6c - 4d + e.$$

Now divide each of the equations marked thus, *, by the coefficient of its firſt term, and there will come out the very values of A, B, C, D, &c. above exhibited. Q. E. D.

COROLLARY.

If every term of the propoſed ſeries a, b, c, d, &c. be ſubtracted from the next following, the firſt of the remainders, $-a + b$, $-b + c$, $-c + d$, $-d + e$, &c. divided by 2, gives the value of B, the coefficient of the ſecond term of the aſſumed ſeries. And, if each of the quantities thus ariſing be ſubtracted from its ſucceeding one, the firſt of the new remainders, $a - 2b + c$, $b - 2c + d$, $c - 2d + e$, &c. divided by 6, will be equal to C, the coefficient of the third term of the ſame ſeries. In like manner, if each of theſe laſt remainders be, again, ſubtracted from its ſucceeding one, the next remainders will be, $-a + 3b - 3c + d$, $-b + 3c - 3d + e$, &c. whereof the firſt, divided by 24, gives the coefficient of the fourth term, &c. &c. Therefore, if the firſt remainder of the firſt order be denoted by P, the firſt of the ſecond order by Q, the firſt of the third by R, the firſt of the fourth by S, &c. then, $\frac{P}{2}$ being $= B$, $\frac{Q}{2 \cdot 3} = C$, $\frac{R}{2 \cdot 3 \cdot 4} = D$, $\frac{S}{2 \cdot 3 \cdot 4 \cdot 5} = E$, &c. it is manifeſt that the ſum of the ſeries $a + b + c + d + e + f$ &c. will be truly expreſſed by

$$an + P \times \frac{n \times \overline{n-1}}{1 \cdot 2} + Q \times \frac{n \times \overline{n-1} \times \overline{n-2}}{1 \cdot 2 \cdot 3} +$$

$$R \times \frac{n \times \overline{n-1} \times \overline{n-2} \times \overline{n-3}}{1 \cdot 2 \cdot 3 \cdot 4} +$$

$$S \times \frac{n \times \overline{n-1} \times \overline{n-2} \times \overline{n-3} \times \overline{n-4}}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}, \&c.$$

Example 1. Let the ſum of the ſeries of ſquares $1 + 4 + 9 + 16 \ldots\ldots + n^2$ be required. Then, taking the differences of the ſeveral orders, according to the preceding corollary, we have

1, 4, 9, 16, 25, 36, &c.
3, 5, 7, 9, 11, &c.
2, 2, 2, 2, &c.
0, 0, 0, &c.

Therefore, a in this caſe being $= 1$, $P = 3$, $Q = 2$, and R, S, *&c.* each $= 0$, the ſum of the whole ſeries, $1 + 4 + 9 + 16 + 25 \ldots\ldots n^2$, is found $= n + \frac{3n \times \overline{n-1}}{2} + \frac{n \times \overline{n-1} \times \overline{n-2}}{3} = \frac{2n^3 + 3n^2 + n}{6} = \frac{n \times \overline{n+1} \times \overline{2n+1}}{6}$.

Example 2. Let it be required to find the ſum of n terms of the following ſeries of cubes, *viz.* $27 + 64 + 125 + 216 + 343 + 512$, *&c.* Proceeding here, as in the laſt example, we have

27, 64, 125, 216, 343, 512, &c.
37, 61, 91, 127, 169, &c.
24, 30, 36, 42, &c.
6, 6, 6, &c.
0, 0, &c.

Therefore, by ſubſtituting 27 for a, 37 for P, 24 for C, and 6 for D, we thence get

$$27n + \frac{37n \times \overline{n-1}}{2} + \frac{24n \times \overline{n-1} \times \overline{n-2}}{2 \cdot 3} + \frac{6n \times \overline{n-1} \times \overline{n-2} \times \overline{n-3}}{1 \cdot 2 \cdot 3 \cdot 4};$$

which, abbreviated, becomes $\frac{n^4}{4} + \frac{5n^3}{2} + \frac{37n^2}{4} + 15n$, the ſum, or value required.

Example 3. Let the ſeries propounded be $2 + 6 + 12 + 20 + 30$, *&c.* In this caſe, we have

2, 6, 12, 20, 30, &c.
4, 6, 8, 10, &c.
2, 2, 2, &c.

Hence, a being $= 2$, P $= 4$, Q $= 2$, and R, S, &c. each $= 0$, the ſum of the ſeries will therefore be

$$2n + \frac{4n \times \overline{n-1}}{2} + \frac{2n \times \overline{n-1} \times \overline{n-2}}{6} = \frac{n^3 + 3n^2 + 2n}{3}$$

$$= \frac{n \times \overline{n+1} \times \overline{n+2}}{3}.$$

And in the very ſame manner the ſum of the ſeries may be truly found, in all caſes where the differences of any order become equal among themſelves: and even in other caſes, where the differences do not terminate, a near approximation may be obtained, by carrying on the proceſs to a ſufficient length.

SECTION XV.

Of Figurate numbers, their ſums, and the ſums of their reciprocals, with other matters of the like nature.

THAT ſeries which ariſes by adding together a rank

of	is called a ſeries of fig. numb. of the
Units (called Fig. N° of the 1ſt ord.)	2d
Figurate numbers of the 2d order	3d
Figurate numbers of the 3d order	4th
Figurate numbers of the 4th order	5th
Figurate numbers of the 5th order	6th
Figurate numbers of the 6th order	7th

order.

Therefore the figurate numbers

of the	are
1ſt order	1 . 1 . 1 . 1 . 1 . &c.
2d order	1 . 2 . 3 . 4 . 5 . &c.
3d order	1 . 3 . 6 . 10 . 15 . &c.
4th order	1 . 4 . 10 . 20 . 35 . &c.
5th order	1 . 5 . 15 . 35 . 70 . &c.

Hence it is manifeſt, that, to find a general expreſſion for a figurate number of any order, is the ſame thing as to find the ſum of all the figurate numbers of the preceding order, ſo far. Let n be put to denote the diſtance of any ſuch number from the beginning of its reſpective order, or the number of terms in the preceding order whereof it is compoſed; then it is evident, by inſpection, that the ſum of the firſt order, or the nth term of the ſecond, will be truly expreſſed by n, the number of terms from the beginning. It is alſo evident, from Sect. 14. p. 203, that the ſum of the ſecond order, $1 + 2 + 3 + 4 \ldots n$, will be $\frac{n^2}{2} + \frac{n}{2}\left(= \frac{n}{1} \times \frac{n+1}{2}\right)$ which, according to the preceding obſervation, is alſo the value of the nth term of the third order. Hence, if the numbers, 1, 2, 3, 4, 5, &c. be ſucceſſively wrote inſtead of n, in the general expreſſion $\frac{n^2}{2} + \frac{n}{2}$, we ſhall thence have $\frac{1}{2} + \frac{1}{2}$, $\frac{4}{2} + \frac{2}{2}$, $\frac{9}{2} + \frac{3}{2}$, $\frac{16}{2} + \frac{4}{2}$, $\frac{25}{2} + \frac{5}{2}$, &c. for the values of the firſt, ſecond, third, fourth, fifth, &c. terms of this order, reſpectively; whence it appears, that the ſeries $1 + 3 + 6 + 10 + 15 + 21$, &c. may be reſolved into theſe two others, *viz.*

$$\frac{1}{2} + \frac{4}{2} + \frac{9}{2} + \frac{61}{2} + \frac{25}{2} + \frac{36}{2} \text{ \&c. and}$$
$$\frac{1}{2} + \frac{2}{2} + \frac{3}{2} + \frac{4}{2} + \frac{5}{2} + \frac{6}{2} \text{ \&c.}$$

The former of which being a ſeries of ſquares, its ſum will therefore be $= \frac{n^3}{6} + \frac{n^2}{4} + \frac{n}{12}$ (*by caſe* 2. *p.* 203) and *that* of the latter ſeries (*by caſe* 1. *p.* 203) appears to be $\frac{n^2}{4} + \frac{n}{4}$: and the aggregate of both, which is $\frac{n^3}{6} + \frac{n^2}{2} + \frac{n}{3}$ (or $\frac{n}{1} \times \frac{n+1}{2} \times \frac{n+2}{3}$) will be the true value of the propoſed ſeries $1 + 3 + 6 + 10 + 15$ &c. continued to n terms, and therefore equal, likewiſe, to the nth term of the next ſuperior order, $1 + 4 + 10 + 20 + 35$ &c. Let, therefore, 1, 2, 3, 4, 5, &c. (as above) be ſucceſſively wrote for n in this general expreſſion, $\frac{n^3}{6} + \frac{n^2}{2} + \frac{n}{3}$, and it will become $\frac{1}{6} + \frac{1}{2} + \frac{1}{3}$, $\frac{8}{6} + \frac{4}{2} + \frac{2}{3}$, $\frac{27}{6} + \frac{9}{2} + \frac{3}{3}$, $\frac{64}{6} + \frac{16}{2} + \frac{4}{3}$, &c. for the values of the firſt, ſecond, third, fourth, &c. terms

terms of the fourth order reſpectively; whence it appears that the ſeries 1 + 4 + 10 + 20 + 35 &c. may be reſolved into theſe three others, *viz.*

$$\frac{1+8+27+64+125+216\ldots\ldots n^3}{6},$$

$$\frac{1+4+9+16+25+36\ldots\ldots n^2}{2},$$

$$\frac{1+2+3+4+5+6\ldots\ldots n}{3};$$

whereof the ſums are $\frac{n^4}{24}+\frac{n^3}{12}+\frac{n^2}{24}$, $\frac{n^3}{6}+\frac{n^2}{4}+\frac{n}{12}$, and $\frac{n^2}{6}+\frac{n}{6}$ (*by p.* 202, and 203) the aggregate of which, or $\frac{n^4}{24}+\frac{n^3}{4}+\frac{11n^2}{24}+\frac{n}{4}$ $\left(=\frac{n}{1}\times\frac{n+1}{2}\times\frac{n+2}{3}\times\frac{n+3}{4}\right)$ will conſequently be the true value of the whole ſeries. After the ſame manner the ſum of the fifth order will appear to be $\frac{n}{1}\times\frac{n+1}{2}\times\frac{n+2}{3}\times\frac{n+3}{4}\times\frac{n+4}{5}$; from whence the law of continuation is manifeſt. And it may not be amiſs to obſerve here, that, though the concluſions thus brought out, are derived by means of the ſums of powers determined in the preceding ſection, yet the ſame values may be otherwiſe obtained, by a direct inveſtigation, from either of the two general methods there laid down.

In order now to find the ſum of the reciprocals of any ſeries of figurate numbers, ſuppoſe $1+b+bc+bcd+bcde+bcdef+$ &c. to be a ſeries whoſe terms continually decreaſe, from the firſt to the laſt, ſo that the laſt may vaniſh, or become indefinitely ſmall: then, by taking the exceſs of every term above that next following one, we ſhall have $1-b$, $b\times\overline{1-c}$, $bc\times\overline{1-d}$, $bcd\times\overline{1-e}$, $bcde\times\overline{1-f}$, &c. The ſum of all which is, evidently, equal to the exceſs of the firſt term above the laſt, or equal to the firſt term, *barely*; becauſe the laſt is ſuppoſed to vaniſh, or to be indefinitely ſmall in reſpect of the firſt. Hence it appears that $\overline{1-b} + b\times\overline{1-c} + bc\times\overline{1-d} + bcd\times\overline{1-e} + bcde\times\overline{1-f}$ &c. $= 1$.

Let b be now taken $=\frac{m}{a}$, $c=\frac{m+p}{a+p}$, $d=\frac{m+q}{a+q}$, $e=\frac{m+r}{a+r}$, $f=\frac{m+s}{a+s}$, &c. Then, $1-b$ being $=\frac{a-m}{a}$, $1-c=\frac{a-m}{a+p}$, $1-d=\frac{a-m}{a+q}$, $1-e=\frac{a-m}{a+r}$, &c. we ſhall, by ſubſtituting theſe ſeveral values in the above equation, have $\frac{a-m}{a} + \frac{m}{a}\times\frac{a-m}{a+p} + \frac{m}{a}\times\frac{m+p}{a+p}\times\frac{a-m}{a+q} + \frac{m}{a}\times\frac{m+p}{a+p}\times\frac{m+q}{a+q}\times\frac{a-m}{a+r}$ + &c. $= 1$; and conſequently $1+\frac{m}{a+p}+\frac{m}{a+p}\times\frac{m+p}{a+q}+\frac{m}{a+p}\times\frac{m+p}{a+q}\times\frac{m+q}{a+r}$ + &c. $=\frac{a}{a-m}$; by dividing the whole by $\frac{a-m}{a}$.

Hence, if q be taken $= 2p$, $r = 3p$, $s = 4p$, &c. and β be put $= a + p$, we ſhall have $1+\frac{m}{\beta}+\frac{m.\overline{m+p}}{\beta.\overline{\beta+p}}+\frac{m.\overline{m+p}.\overline{m+2p}}{\beta.\overline{\beta+p}.\overline{\beta+2p}}+\frac{m.\overline{m+p}.\overline{m+2p}.\overline{m+3p}}{\beta.\overline{\beta+p}.\overline{\beta+2p}.\overline{\beta+3p}}$ + &c. *ad infinitum*, $=\frac{\beta-p}{\beta-p-m}$; which, when $p=1$, becomes $1+\frac{m}{\beta}+\frac{m.\overline{m+1}}{\beta.\overline{\beta+1}}+\frac{m.\overline{m+1}.\overline{m+2}}{\beta.\overline{\beta+1}.\overline{\beta+2}}$ + &c. $=\frac{\beta-1}{\beta-m-1}$: this, by taking $m = 1$ and $\beta = n$, gives $1+\frac{1}{n}+\frac{1.2}{n.\overline{n+1}}+\frac{1.2.3}{n.\overline{n+1}.\overline{n+2}}+$

$$+\frac{1\,.\,2\,.\,3\,.\,4}{n\,.\,\overline{n+1}\,.\,\overline{n+2}\,.\,\overline{n+3}}+\&c.=\frac{n-1}{n-2};$$ exhibiting the general value of a ſeries of the reciprocals of figurate numbers, infinitely continued; whereof the order is repreſented by n: from whence as many particular values as you pleaſe may be determined. Thus, by expounding n by 3, 4, 5, &c. ſucceſſively, it appears that

$$1+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+\frac{1}{15}\ \&c.=2,$$

$$1+\frac{1}{4}+\frac{1}{10}+\frac{1}{20}+\frac{1}{35}\ \&c.=\frac{3}{2},$$

$$1+\frac{1}{5}+\frac{1}{15}+\frac{1}{35}+\frac{1}{70}\ \&c.=\frac{4}{3}.$$

And ſo on, for any higher order; but the ſums of the two firſt, or loweſt orders cannot be determined, theſe being infinite.

By interpreting β and m by different values, the ſums of various other ſeries's may be deduced from the ſame general equation. Thus, in the firſt place, let $\beta = m+2$; ſo ſhall the ſaid equation become $1+\frac{m}{m+2}+$

$$\frac{m\,.\,\overline{m+1}}{\overline{m+2}\,.\,\overline{m+3}}+\frac{m\,.\,\overline{m+1}}{\overline{m+3}\,.\,\overline{m+4}}+\frac{m\,.\,\overline{m+1}}{\overline{m+4}\,.\,\overline{m+5}}\ \&c.$$

$= m+1$; which, divided by $m\,.\,\overline{m+1}$, gives

$$\frac{1}{m\,.\,\overline{m+1}}+\frac{1}{\overline{m+1}\,.\,\overline{m+2}}+\frac{1}{\overline{m+2}\,.\,\overline{m+3}}+\frac{1}{\overline{m+3}\,.\,\overline{m+4}}$$

$$\&c.=\frac{1}{m}.$$

Again, by taking $\beta = m+3$, and dividing the whole equation by $m\,.\,\overline{m+1}\,.\,\overline{m+2}$, we have

$$\frac{1}{m\,.\,\overline{m+1}\,.\,\overline{m+2}}+\frac{1}{\overline{m+1}\,.\,\overline{m+2}\,.\,\overline{m+3}}+$$

$$\frac{1}{\overline{m+2}\,.\,\overline{m+3}\,.\,\overline{m+4}}\ \&c.=\frac{1}{m\,.\,\overline{m+1}\,.\,2}.$$

In like manner we ſhall have $\frac{1}{m.\overline{m+1}.\overline{m+2}.\overline{m+3}} + \frac{1}{\overline{m+1}.\overline{m+2}.\overline{m+3}.\overline{m+4}}$, &c. $= \frac{1}{m.\overline{m+1}.\overline{m+2}.3}$. From whence the law for continuing the ſums of theſe laſt kinds of ſeries's is manifeſt; by which it appears, that, if inſtead of the laſt factor in the denominator of the firſt term, the exceſs thereof above the firſt factor be ſubſtituted, the fraction thence ariſing will truly expreſs the value of the whole infinite ſeries.

A few other particular caſes will further ſhew the uſe of the general equations above exhibited.

Let the ſum of the ſeries $1 + \frac{2}{5} + \frac{2}{5} \times \frac{4}{7} + \frac{2}{5} \times \frac{4}{7} \times \frac{6}{9} + \frac{2}{5} \times \frac{4}{7} \times \frac{6}{9} \times \frac{8}{11}$ &c. *ad infinitum*, be required.

Here, by comparing the propoſed ſeries with $1 + \frac{m}{\beta} + \frac{m.\overline{m+p}}{\beta.\overline{\beta+p}} +$ &c. $(= \frac{\beta - p}{\beta - p - m})$ we have $m = 2$, $\beta = 5$, and $p = 2$; and conſequently $\frac{\beta - p}{\beta - p - m} = 3$ = the true value of the ſeries.

Let the ſum of an infinite ſeries of this form, *viz.* $\frac{1}{1.2.3\,\&c.} + \frac{1}{2.3.4\,\&c.} + \frac{1}{3.4.5\,\&c.} +$ &c. be demanded.

Here (according to the preceding rule) we have

$$\frac{1}{1\,.\,2} + \frac{1}{2\,.\,3} + \frac{1}{3\,.\,4}\ \&c. = \frac{1}{1\,.\,1} = 1;$$

$$\frac{1}{1.2.3} + \frac{1}{2.3.4} + \frac{1}{3.4.5}\ \&c. = \frac{1}{1.2.2} = \frac{1}{4};$$

$$\frac{1}{1.2.3.4} + \frac{1}{2.3.4.5} + \frac{1}{3.4.5.6}\ \&c. = \frac{1}{1.2.3.3} = \frac{1}{18};$$

&c. &c.

If, inſtead of the whole infinite ſeries, you want the ſum of a given number of the leading terms, only; then let the value of the remaining part be found, *as above*, and ſubtracted from the whole, and you will have your deſire.

Thus, for inſtance, let it be required to find the ſum of the ten firſt terms of the ſeries $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5}$ &c. Then the remaining part, $\frac{1}{11 \cdot 12} + \frac{1}{12 \cdot 13} + \frac{1}{13 \cdot 14} + \frac{1}{14 \cdot 15}$ &c. being $= \frac{1}{11}$ (*by the rule above*) and the whole ſeries $= 1$, the value here ſought will therefore be $1 - \frac{1}{11} = \frac{10}{11}$. The like of others.

The ſums of ſeries's ariſing from the multiplication of the terms of a rank of figurate numbers into thoſe of a decreaſing geometrical progreſſion, are deduced in the following manner.

By the theorem for involving a binomial (given at p. 40. and demonſtrated hereafter) it is known that $\frac{1}{\overline{1-x}|^{m}}$ (or $\overline{1-x}|^{-m}$) is $= 1 + mx + m \cdot \frac{m+1}{2} \cdot x^2 + m \cdot \frac{m+1}{2} \cdot \frac{m+2}{3} \cdot x^3 + m \cdot \frac{m+1}{2} \cdot \frac{m+2}{3} \cdot \frac{m+3}{4} \cdot x^4$ &c. In which equation let m be expounded by 1, 2, 3, 4, 5, &c. ſucceſſively, ſo ſhall

1°. $\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + x^5 +$ &c.

2°. $\frac{1}{\overline{1-x}|^2} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5$ &c.

3°. $\frac{1}{\overline{1-x}|^3} = 1 + 3x + 6x^2 + 10x^3 + 15x^4 + 21x^5$ &c.

4°. $\frac{1}{\overline{1-x}|^4} = 1 + 4x + 10x^2 + 20x^3 + 35x^4 + 56x^5$ &c.

5°.

5°. $\frac{1}{\overline{1-x}|^{5}} = 1 + 5x + 15x^2 + 35x^3 + 70x^4 + 126^5$ &c.

6°. $\frac{1}{\overline{1-x}|^{6}} = 1 + 6x + 21x^2 + 56x^3 + 126x^4 + 252x^5$ &c.

All which ſeries's (whereof the ſums are thus given) are ranks of the different orders of figurate numbers, multiplied by the terms of the geometrical progreſſion $1, x, x^2, x^3, x^4,$ &c.

From theſe equations the ſums of ſeries's compoſed of the terms of a rank of powers, drawn into thoſe of a geometrical progreſſion, ſuch as $1 + 4x + 9x^2 + 16x^3$ &c. and $1 + 8x + 27x^2 + 64x^3$ &c. may alſo be derived; there being, as appears from the former part of this ſection, a certain relation between the terms of a ſeries of powers and thoſe of figurate numbers; the latter being *there* determined by means of the former. To find *here* the converſe relation, or to determine the former from the latter, it will be expedient to multiply the ſeveral equations above brought out, by a certain number of terms of an aſſumed ſeries $1 + Ax + Bx^2 + Cx^3$ &c. in order that the coefficients of the powers of x may, by regulating the values A, B, C, D, *&c.* become the ſame as in the ſeries given.

Thus, if the ſeries given be $1 + 4x + 9x^2 + 16x^3 + 25x^4$ &c.; then, by multiplying our third equation by $1 + Ax$, we ſhall have $\frac{1 + Ax}{\overline{1-x}|^{3}} = 1 + \overline{3 + A} \times x + \overline{6 + 3A} \times x^2 + \overline{10 + 6A} \times x^3 +$ &c. which ſeries, it is evident by inſpection, will be exactly the ſame, in every term, with the propoſed one, if the quantity A be taken $= 1$. The ſum of the ſaid ſeries, infinitely continued, is therefore truly repreſented by $\frac{1 + x}{\overline{1-x}|^{3}}$.

In like manner, if the fourth equation $\frac{1}{\overline{1-x}|^{4}} = 1 + 4x + 10x^2 + 20x^3 + 35x^4$ &c. be multiplied by

1 +

$1+Ax+Bx^2$, there will ariſe $\frac{1+Ax+Bx^2}{\overline{1-x}|^4}=1+\overline{4+A}\times x+\overline{10+4A+B}\times x^2+\overline{20+10A+4B}\times x^3$ &c. where, the ſeveral terms of the ſeries being compared with thoſe of the ſeries $1+8x+27x^2+64x^3$ &c. we have $4+A=8$, and $10+4A+B=27$; whence $A=4$, and $B=1$; and conſequently, by ſubſtituting theſe values, $\frac{1+4x+x^2}{\overline{1-x}|^4}=1+8x+27x^2+64x^3+125x^4$ &c.

Again, by multiplying the fifth equation, $\frac{1}{\overline{1-x}|^5}=1+5x+15x^2+35x^3$ &c. by $1+Ax+Bx^2+Cx^3$, *it* becomes $\frac{1+Ax+Bx^2+Cx^3}{\overline{1-x}|^5}=1+\overline{5+A}\times x+\overline{15+5A+B}\times x^2+\overline{35+15A+5B+C}\times x^3$ &c. And, by comparing the ſeveral terms of the ſeries with thoſe of $1+16x+81x^2+256x^3$ &c. we get $5+A=16$, $15+5A+B=81$, and $35+15A+5B+C=256$: whence $A=11$, $B\ (=81-15-55)=11$, and $C\ (=256-35-220)=1$; and conſequently $\frac{1+11x+11x^2+x^3}{\overline{1-x}|^5}=1+16x+81x^2+256x^3$ &c.

By proceeding the ſame way it will be found, that $\frac{1+26x+66x^2+26x^3+x^4}{\overline{1-x}|^6}=1+2^5x+3^5x^2+4^5x^3+$ &c. &c.

And, *univerſally*, putting $a=m$, $b=m\cdot\frac{m+1}{2}$, $c=m\cdot\frac{m+1}{2}\cdot\frac{m+2}{3}$, &c. and multiplying the general equation, $\frac{1}{\overline{1-x}|^m}=1+ax+bx^2+cx^3+dx^4$ &c. by $1+Ax+Bx^2+Cx^3+Dx^4$ &c. there ariſes $\frac{1+Ax+Bx^2\ \&c.}{\overline{1-x}|^m}=1+\overline{a+A}\times x+$

$+\ b$

$+ \overline{b + aA + B} \times x^2 + \overline{c + bA + aB + C} \times x^3$ &c. The terms of which ſeries being compared with thoſe of the ſeries $1 + 2^n x + 3^n x^2 + 4^n x^3 + 5^n x^4$, &c. we have $A = 2^n - a$, $B = 3^n - aA - b$, $C = 4^n - aB - bA - c$, $D = 5^n - aC - bB - cA - d$, &c. where the law of continuation is manifeſt; and where, from the law obſerved in all the preceding caſes, it appears, that the value of m muſt exceed the index n, of the given ſeries of powers, by an unit; and that the ſeries $1 + Ax + Bx^2 + Cx^3$ &c. will always conſiſt of n terms; whereof the coefficients of the firſt and laſt, the ſecond and laſt but one, *&c.* will be reſpectively equal to each other: ſo that, having found from the preceding equations as many of the quantities A, B, C *&c.* as are expreſſed by $\frac{1}{2}n - 1$, the others will be given from thence, and conſequently, $\dfrac{1 + Ax + Bx^2 + Cx^3 \text{ \&c.}}{\overline{1 - x}|^{n+1}}$ the true value of the propoſed ſeries $1 + 2^n x + 3^n x^2 + 4^n x^3$ &c. Thus for example, let $n = 6$: then $m = 7 = a$, $b = 28$, $A = 64 - 7 = 57$, $B = 729 - 399 - 28 = 302$; and therefore

$$\frac{1 + 57x + 302x^2 + 302x^3 + 57x^4 + x^5}{\overline{1 - x}|^7} = 1 + 2^6 x + 3^6 x^2 + 4^6 x^3 \text{ \&c.}$$

and ſo of others.

Theſe equations, or theorems, give the ſum of the whole ſeries, infinitely continued; but from thence the ſum of any aſſigned number of terms may be determined, not only when the coefficients are a ſeries of powers, but likewiſe when they are produced by factors that are unequal: the method of which I ſhall inſtance in finding the ſum of t terms of the ſeries $\overline{f - p} \cdot \overline{g - q} \cdot z^r + \overline{f - 2p} \cdot \overline{g - 2q} \cdot z^{r+v} + \overline{f - 3p} \cdot \overline{g - 3q} \cdot z^{r+2v} +$ &c. Which ſeries, by actually multiplying the factors together, is reſolved into the three following ones.

$$fgz^r \times \overline{1 + z^v + z^{2v} + z^{3v} + z^{4v}} \text{ \&c.}$$
$$- \overline{fq + gp} \cdot z^r \times \overline{1 + 2z^v + 3z^{2v} + 4z^{3v}} \text{ \&c.}$$
$$+ pqz^r \times \overline{1 + 4z^v + 9z^{2v} + 16z^{3v}} \text{ \&c.}$$

The ſum of the firſt of theſe, infinitely continued, ſuppoſing $x = z^v$, will be $= \frac{fgz^r}{1 - x}$; that of the ſecond $= - \frac{\overline{fq + gp} \cdot z^r}{\overline{1 - x}|^2}$; and that of the third $= \frac{\overline{1 + x} \cdot pqz^r}{\overline{1 - x}|^3}$, *by what has been above determined*; and conſequently the ſum of all the three equal to $\frac{z^r}{1 - x} \times \overline{fg - \frac{fq + gp}{1 - x} + \frac{pq \cdot \overline{1 + x}}{\overline{1 - x}|^2}}$ = the whole infinite ſeries $\overline{f - p} \cdot \overline{g - q} \cdot z^r + \overline{f - 2p} \cdot \overline{g - 2q} \cdot z^{r+v} +$ *&c.* But the ſum of the t firſt terms only is wanted; therefore the ſum of all the remaining terms, after the t firſt, muſt be found in like manner, and be deducted from the ſum of the whole, here given. Now, to do this, we are firſt to get the leading term of the ſaid remaining ones; which, according to the law of the ſeries, will be expreſſed by $\overline{f - p - tp} \cdot \overline{g - q - tq} \cdot z^{r+tv}$: whence, if we make $f - tp = h$, $g - tq = k$, and $r + tv = s$, it is evident, that the ſeries to be deducted will be $\overline{h - p} \cdot \overline{k - q} \cdot z^s + \overline{h - 2p} \cdot \overline{k - 2q} \cdot z^{s+v}$ &c. which having the very ſame form with that firſt propoſed, it's ſum will therefore be had by, barely, writing h for f, k for g, and s for r, in the value above determined: which, thereby, becomes

$$\frac{z^s}{1 - x} \times \overline{hk - \frac{hq + kp}{1 - x} + \frac{pq \cdot \overline{1 + x}}{\overline{1 - x}|^2}}.$$

In the ſame manner, ſuppoſing the t firſt terms of the ſeries $\overline{a-p}\,.\,\overline{b-p}\,.\,\overline{c-p}\,.\,\overline{d-p}\,.$ &c. $\times z^{r}+\overline{a-2p}\,.\,\overline{b-2p}\,.\,\overline{c-2p}\,.\,\overline{d-2p}\,.$ &c. $\times z^{r+v}$ &c. were to be required; by putting the continual product of all the quantities a, b, c, d, &c. $=P$; the ſum of all the products ($\frac{P}{a}+\frac{P}{b}+\frac{P}{c}$ &c.) that ariſe by omitting one letter in each, $=Q$; the ſum of all thoſe ($\frac{P}{ab}+\frac{P}{ac}$ &c.) by omitting two letters, $=R$, &c. we ſhall *here* have

$$\frac{z^{r}}{1-x}\times\left\{\overline{P-\frac{Qp}{1-x}+\frac{Rp^{2}\,.\,\overline{1+x}}{\overline{1-x}|^{2}}-\frac{Sp^{3}\,.\,\overline{1+4x+x^{2}}}{\overline{1-x}|^{3}}+\frac{Tp^{4}\,.\,\overline{1+11x+11x^{2}+x^{3}}}{\overline{1-x}|^{4}}}\right.,$$

&c. for the ſum of the whole infinite ſeries: and, if we make $\acute{a}=a-tp\,.\,\acute{b}=b-tp$, $\acute{r}=r+tv$, &c. it is evident that the ſum of the remaining terms, after the t firſt, will be truly expreſſed by

$$\frac{z^{r}}{1-x}\times\overline{\acute{P}-\frac{\acute{Q}p}{1-x}+\frac{\acute{R}p^{2}\,.\,\overline{1+x}}{\overline{1-x}|^{2}}-\frac{\acute{S}p^{3}\,.\,\overline{1+4x+x^{2}}}{\overline{1-x}|^{3}}}$$

&c. where $x=z^{v}$, and $\acute{P}$, $\acute{Q}$, $\acute{R}$, $\acute{S}$, &c. are the ſame in relation to $\acute{a}$, $\acute{b}$, $\acute{c}$, $\acute{d}$, &c. as P, Q, R, S, &c. in reſpect to a, b, c, d, &c.

A multitude of other caſes and examples might be given, there not being, in the whole ſcope of the mathematical ſciences, a ſubject of greater variety and intricacy than this buſineſs of ſeries's: but to purſue *it* farther *here* would be inconſiſtent with the general plan of this work. Such, therefore, who are deſirous of a greater inſight into the matter, may, if they pleaſe, turn to my Miſcellanies, where *it* is carried to a greater length.

From the ſeries's for figurate numbers, derived in the former part of this ſection, the inveſtigation of a general theorem for determining how many different combinations any number of things will admit of, when taken two by two, three by three, &c. may be be very eaſily deduced. *Let the number of things in each combination be, firſt, ſuppoſed two, only; and let* n *be, univerſally, put to repreſent the whole number of things, or letters,* a, b, c, d, &c. *to be combined.* When the number of things is only two, as a and b, it is evident that there can be only one combination (ab); but, if n be increaſed by 1, or the letters to be combined be three, as a, b, c, then it is plain that the number of combinations will be increaſed by 2, the number of the preceding letters a and b; ſince, with each of *thoſe*, the new letter c may be joined; and therefore the whole number of combinations, in this caſe, will be truly expreſſed by $1 + 2$. Again, if n be increaſed by one more, or the whole number of letters be four, as a, b, c, d; then it will appear that the number of combinations muſt be increaſed by 3, ſince 3 is the number of the preceding letters, with which the new letter d can be combined, and therefore will, here, be truly expounded, by $1 + 2 + 3$. And, by reaſoning in the ſame manner, it will appear, that the whole number of combinations of two, in five things, will be $1 + 2 + 3 + 4$; in ſix things, $1 + 2 + 3 + 4 + 5$; and in ſeven, $1 + 2 + 3 + 4 + 5 + 6$; &c. Whence, univerſally, the number of combinations of n things, taken two by two, is $= 1 + 2 + 3 + 4 + \ldots\ldots n - 1$: which being a ſeries of figurate numbers of the ſecond order, where the number of terms is $n - 1$, the ſum thereof, by caſe 1, p. 203, will therefore be truly defined by $\frac{n-1}{1} \times \frac{n}{2}$, or $n \times \frac{n-1}{2}$.

Let now the number of quantities in each combination be ſuppoſed to be three.

It is plain, that, in three things, a, b, c, there can be only one combination; but, if n be increaſed by 1, or the number of things be 4, as *a*, *b*, *c*, *d*, then will the number of combinations be increaſed by (3) the number of all the combinations of two, in the preceding letters *a*, *b*, *c*; ſince with each two of thoſe the new letter *d* may be combined; therefore the number of combinations, in this caſe, is 1 + 3. Again, if n be ſuppoſed to be increaſed by 1 more, or the number of letters to become five, as *a*, *b*, *c*, *d*, *e*; then the number of combinations will be increaſed by ſix more (= 1 + 2 + 3), that is, by all the combinations of two, in the four preceding letters, *a*, *b*, *c*, *d*; ſince, (as before) with each two of thoſe, the new letter *e* may be combined. Hence the number of combinations of n things, taken three by three, appears to be 1 + 3 + 6 + 10 &c. continued to $n-2$ terms; which being a ſeries of figurate numbers of the third order, the value thereof, by what is before determined (p. 214) will be truly expreſſed by $\frac{n-2}{1}\times\frac{n-1}{2}\times\frac{n}{3}$, or, its equal, $\frac{n}{1}\times\frac{n-1}{2}\times\frac{n-2}{3}$.

And univerſally, ſince it appears, that increaſing the number of letters by 1, always increaſes the number of combinations by all the combinations of the next inferior order with the preceding letters (for this obvious reaſon, that to each of theſe laſt combinations, the new letter may be joined), it is manifeſt, that the combinations, of any order, obſerve the ſame law, and are generated in the very ſame manner as figurate numbers; and therefore may be exhibited, by the ſame general expreſſions; only, as there are 2, 3, 4, 5, &c. things neceſſary to form the firſt, or one ſingle combination, according to the different caſes, it is plain, that the number of terms muſt be leſs by 1, 2, 3, &c. reſpectively, than (n) the number of things; and therefore, inſtead of n, in the aforeſaid general expreſſions, we muſt ſubſtitute $n-1$, $n-2$, or $n-3$, &c. reſpectively, to have the true value *here*. Hence, the number of combinations of two things, in n things, will be $\frac{n-1}{1}\times\frac{n}{2}$, or $\frac{n}{1}\times\frac{n-1}{2}$; of three, $\frac{n-2}{1}\times\frac{n-1}{2}\times\frac{n}{3}$, or $\frac{n}{1}\times\frac{n-1}{2}\times\frac{n-2}{3}$; of four, $\frac{n-3}{1}\times\frac{n-2}{2}\times\frac{n-1}{3}\times\frac{n}{4}$, or $\frac{n}{1}\times\frac{n-1}{2}\times\frac{n-2}{3}\times\frac{n-3}{4}$ (*vid. p.* 215): whence, univerſally, the number of combinations in the number, n, of things, taken two by two, three by three, *&c.* will be expreſſed by $\frac{n}{1}\times\frac{n-1}{2}\times\frac{n-2}{3}\times\frac{n-3}{4}$ *&c.* continued to as many factors as there are things in each combination.

From this laſt general expreſſion, ſhewing the combinations which any number of quantities will admit of, the known theorem for raiſing a binomial, to any given power, is very eaſily, and naturally derived. For, it is plain that $aa\left.\begin{matrix}+b\\+c\end{matrix}\right\}a+bc=\overline{a+b}\times\overline{a+c}$; which, multiplied by $a+d$, gives $a^3\left.\begin{matrix}+b\\+c\\+d\end{matrix}\right\}a^2+\left.\begin{matrix}bc\\bd\\cd\end{matrix}\right\}a+bcd=\overline{a+b}\times\overline{a+c}\times\overline{a+d}$; and this, again, multiplied by $a+e$, gives

$$a^4+\left.\begin{matrix}+b\\+c\\+d\\+e\end{matrix}\right\}a^3+\left.\begin{matrix}+bc\\+bd\\+be\\+cd\\+ce\\+de\end{matrix}\right\}a^2+\left.\begin{matrix}+bcd\\+bce\\+bde\\+cde\end{matrix}\right\}a+bcde=\overline{a+b}\times\overline{a+c}\times\overline{a+d}\times\overline{a+e}.$$

Whence it appears, that the coefficient of a, in the ſecond term, is always the ſum of all the other quantities b, c, d, &c. added together; and that the coefficient of the third term is the ſum of all the products of thoſe quantities, or of all their poſſible combinations, taken two by two; ſince, from the nature of multiplication, they muſt be all concerned alike, in every term: whence it is alſo manifeſt, that the coefficient of the fourth term muſt be the ſum of all the ſolids of the ſame quantities, or of all their poſſible combinations, taken three by three, *&c. &c.*

Hence, if the number of the quantities b, c, d, e, &c. or the number of the factors, $a+b$, $a+c$, $a+d$, $a+e$, &c. to be multiplied continually together, be denoted by n; it follows, that the number of letters, or quantities in the coefficient of the ſecond term of the product will likewiſe be denoted by n; that the number of all their products, or of all the combinations of two, in the coefficient of the third term, will be $n \times \frac{n-1}{2}$ (it having been ſhewn above that the number of combinations of n things, taken two by two, is $n \times \frac{n-1}{2}$); and that the number of all the ſolids of thoſe quantities, or all the combinations of three, in the coefficient of the fourth term, will be $n \times \frac{n-1}{2} \times \frac{n-2}{3}$, &c. Therefore, if all the quantities b, c, d, e, &c. be now taken equal to each other, it is evident, that $\overline{a+b} \times \overline{a+c} \times \overline{a+d} \times \overline{a+e}$ &c. will become $\overline{a+b} \times \overline{a+b} \times \overline{a+b} \times \overline{a+b}$ &c. or $\overline{a+b}\rceil^{n}$; and that the coefficient of the power of a, in the ſecond term of the product, will be nb; in the third $n \times \frac{n-1}{2} b^2$ (ſince all the rectangles, as well as all the ſolids, &c. do here become equal); and in the fourth $n \times \frac{n-1}{2} \times \frac{n-2}{3} b^3$ &c. But it is evident, from the nature of multiplication, that the powers of a, in the ſecond, third, fourth, &c. terms of $a+b$ raiſed to the power n, are a^{n-1}, a^{n-2}, a^{n-3}, &c. Therefore $\overline{a+b}\rceil^{n}$, or $a+b$ raiſed to the power n, is truly expreſſed by $a^n + nba^{n-1} + n \times \frac{n-1}{2} b^2 a^{n-2} + n \times \frac{n-1}{2} \times \frac{n-2}{3} b^3 a^{n-3}$ &c. or $a^n + na^{n-1} b + n \times \frac{n-1}{2} a^{n-2} b^2 + n \times \frac{n-1}{2} \times \frac{n-2}{3} a^{n-3} b^3$ &c. as was to be ſhewn.

SECTION XVI.

Of Intereſt and Annuities.

INTEREST may be either ſimple or compound: ſimple intereſt is that which is paid for the loan of any principal, or ſum of money, lent out for ſome limited time, at a certain rate *per cent.* agreed upon between the borrower and the lender, and is always proportional to the time. Thus, if the rate agreed upon, be 4 *per cent. per annum*, or, which is the ſame, if the intereſt of 100 *l.* for one year be 4*l.* then the ſimple intereſt of the ſame ſum for two years, will be 8 *l.* for three years 12 *l.* and for four years 16 *l.* and ſo on for any other time in proportion.

Compound intereſt is that which ariſes by leaving the ſimple intereſt of any ſum of money, after it becomes due, together with the principal, in the hands of the borrower, and thereby converting the whole into a new principal. Thus, he who lets out 100 *l.* for one year, at the rate of 4 *per cent.* has a right to receive 104*l.* at the year's end; which ſum he may leave in the borrower's hands, a ſecond year, as a new principal, in order to receive intereſt for the whole; and this intereſt (which will be found 4*l.* 3*s.* $2\frac{1}{2}$*d.*) together with 4*l.* the intereſt of the firſt principal, for the firſt year, will be the compound intereſt of 100 *l.* for two years: and ſo on, for any greater number of years. But I ſhall firſt give the inveſtigation of the theorems for ſimple intereſt.

Let the rate *per cent.* or the intereſt of 100*l.* for one year $= r$; the months, weeks, or days in one year $= t$; the months, weeks, or days which any ſum, a, is lent out for $= n$; and the amount of that ſum, in the ſaid time, *viz.* principal and intereſt, $= b$.

Then it will be as 100 is to r (the intereſt of 100 *l.*) ſo is the propoſed ſum (a) to $\frac{ar}{100}$, the intereſt of that ſum, for the ſame time. Again, as t, the time in which the ſaid intereſt is produced, is to n (the time propoſed) ſo is $\frac{ar}{100}$, the intereſt in the former of theſe times, to $\frac{anr}{100t}$, that in the latter; which, added to a, the principal, gives $a + \frac{anr}{100t} = b$ (the whole amount): from whence, we alſo have $a = \frac{100bt}{100t + nr}$, $r = \frac{100t \times \overline{b-a}}{an}$, and $n = \frac{100t \times \overline{b-a}}{ar}$: the uſe of which equations, or theorems, will appear by the following examples.

Examp. 1. What is the amount of 550 *l.* at 4 *per cent.* in ſeven months?

In this caſe we have $a = 550$, $r = 4$, $t = 12$, $n = 7$; and conſequently $a + \frac{anr}{100t} = 550 + \frac{550 \times 7 \times 4}{100 \times 12} = 562\frac{5}{6}$ *l.* or 562 *l.* 16 *s.* 8 *d.* the true value ſought,

Examp. 2. What is the intereſt of 1 *l.* for one day, at the rate of 5 *per cent.*?

Here r being $= 5$, $t = 365$, $a = 1$, and $n = 1$, we have $\frac{anr}{100t} = \frac{5}{100 \times 365} = \frac{1}{100 \times 73} = 0{,}0001369863$ &c. = the decimal parts of a pound required.

Examp. 3. What ſum, in ready money, is equivalent to 600 *l.* due nine months hence, allowing 5 *per cent.* diſcount?

Here r being $= 5$, $t = 12$, $n = 9$, and $b = 600$, we have a (*by Theorem* 2) $= \frac{100 \times 600 \times 12}{100 \times 12 + 9 \times 5} = 578{,}313$ *l.* or 578 *l.* 6 *s.* $3\frac{1}{4}$ *d.* which is the value required.

Examp. 4. At what rate of intereſt will 300 *l.* in fifteen months, amount to, or raiſe a ſtock of 330 *l.*?

In this caſe we have given $t = 12$, $n = 15$, $a = 300$, and $b = 330$; whence (*by Theorem* 3) r will come out $= \frac{100 \times 12 \times 30}{300 \times 15} = 8$; therefore 8 *per cent.* is the rate required.

Examp. 5. In how many days will 365*l.* at the rate of 4 *per cent.* amount to, or raiſe a ſtock of 400*l.*?

Here (*by Theorem* 4) we have $n = \frac{100 \times 365 \times 35}{365 \times 4}$ $= 875 =$ the number of days required.

Of Annuities or Penſions in arrear, computed at ſimple Intereſt.

Annuities or penſions in arrear are ſuch, which, being payable, or becoming due, yearly, remain unpaid any number of years: and we are to compute what all thoſe payments will amount to, allowing ſimple intereſt for their forbearance, from the time each particular payment becomes due: in order to which,

Let
$A =$ the annuity, penſion, or yearly rent.
$n =$ the time, or number of years, it is forborn.
$r =$ the intereſt of 1 *l.* for one year.
$m =$ the amount of the annuity and its intereſt.

Then, as $1 : r :: A : rA$, the intereſt of the propoſed ſum or penſion A, for one year; which, as the laſt year's rent but one, is forborn only one year; will expreſs the whole intereſt of that rent, or payment: moreover, ſince the laſt year's rent but two is forborn two years, its intereſt will be $2rA$: and, in the ſame manner, that of the laſt year's rent but three, will appear to be $3rA$, &c. &c. whence it is manifeſt that the ſum total of all theſe, or the whole intereſt, to be received at the expiration of n years, for the forbearance of the propoſed annuity or penſion, will be truly defined by the arithmetical progreſſion $rA + 2rA + 3rA + 4rA + 5rA$ &c. continued to $n-1$ terms, that is, to as many terms as there are years, excepting the laſt. But the ſum of this progreſſion is equal to $n \times \frac{n-1}{2} \times rA$ (*by Theor.* 4. *Sect.* 10.) Therefore, if to *this*, the aggregate of all the rents, or nA, be added, we ſhall have $nA + \frac{n \times \overline{n-1}}{2} \times rA = m$:

whence

whence we, alſo, have $A = \dfrac{m}{n + n \times \dfrac{n-1}{2} \times r}$,

$r = \dfrac{2m - 2nA}{n \times \overline{n-1} \times A}$, and $n = \sqrt{\dfrac{2m}{rA} + p^2} - p$; ſuppoſing $p = \dfrac{1}{r} - \dfrac{1}{2}$.

Examp. 1. If 600*l.* yearly rent, or penſion, be forborn five years, what will it amount to, allowing 4 *per cent.* intereſt for each payment, from the time it becomes due?

Here we have given $A = 600$, $n = 5$, and $r = .04$ (for as 100 : 4 : : 1 : .04) which values ſubſtituted in *Theorem* 1, give $m = (nA + n \times \dfrac{n-1}{2} Ar = 3000 + 240) = 3240$*l.* for the value that was to be found.

Examp. 2. What annuity, or yearly penſion, being forborn five years, will, in that time, amount to, or raiſe a ſtock of 3240*l.* at 4 *per cent.* intereſt?

In this caſe we have given $n = 5$, $r = .04$, and $m = 3240$, and therefore, *by Theorem* 2, $A \left(= \dfrac{m}{n + \frac{1}{2}n \times \overline{n-1} \times r} = \dfrac{3240}{5 + .4}\right) = 600$; which is the annuity required.

Examp. 3. At what rate of intereſt will an annuity of 560*l.* in ſeven years, raiſe a ſtock of 4508*l.*?

In this caſe we have given $A = 560$, $n = 7$, and $m = 4508$; whence (*by Theor.* 3) we have $r \left(= \dfrac{2m - 2nA}{n \times \overline{n-1} \times A}\right.$ $\left.= \dfrac{9016 - 5840}{42 \times 560}\right) = .05 =$ the intereſt of 1*l.* for one year; therefore it will be as 1 : ,05 : : 100 : 5 (*per cent.*) the rate required.

Examp. 4. How long muſt an annuity of 560*l.* be forborn, to raiſe a ſtock of 4508*l.* ſuppoſing intereſt to be 5 *per cent.*?

Here, we have given $A = 560$, $r = .05$, $m = 4508$; whence, *by Theorem* 4, we alſo have $p \left(= \frac{1}{r} - \frac{1}{2}\right) = 19.5$; and conſequently $n \left(= \sqrt{\frac{2m}{rA} + p^2} - p\right) = 73$ which is the number of years required.

Note. If the rent or penſion is payable half-yearly, or quarterly, the method of proceeding will be *ſtill* the ſame, provided n be always taken to expreſs the number of payments, and r the intereſt of 1 *l.* for the time in which the firſt payment becomes due. Thus, if it were required to find what 300*l.* half-yearly penſion would amount to in five years at 4 *per cent.* intereſt: then the ſimple intereſt of 1 *l.* for half a year being $= ,02$, and the number of payments $= 10$, we, in this caſe, have $A = 300$, $r = ,02$, and $n = 10$; and conſequently m (*by Theorem* 1) $= rA + n \times \frac{n-1}{2} \times rA = 3270l.$ which is the value ſought. And the like is to be obſerved in what follows hereafter.

Of the preſent values of Annuities, or Penſions, computed at ſimple intereſt.

$$\text{Let}\begin{cases} A = \text{the annuity, penſion, or yearly rent.} \\ r = \text{the intereſt of 1 } l. \text{ for one year.} \\ n = \text{the number of years.} \\ v = \text{the preſent value of the annuity.} \end{cases}$$

Then, becauſe the amount of the annuity, in n years, is found above to be $nA + \frac{1}{2}n\,\overline{n-1}\,.\,rA$, and ſince 1 *l.* preſent money, is equivalent to $1 + nr$ to be received at the end of the time n, we therefore have $1 + nr : 1 :: nA + \frac{1}{2}n\,.\,\overline{n-\quad}\,.\,rA$ (the ſaid amount) $: \frac{nA + \frac{1}{2}n\,.\,\overline{n-1}\,.\,rA}{1 + nr}$, its required value, in preſent money. But it may be obſerved that this method, given by authors for determining the values of annuities, *according to ſimple intereſt*, is, in reality, a particular ſort, or ſpecies of *compound intereſt*: ſince the allowing of intereſt upon the annuity, as it becomes due, is nothing leſs than allowing intereſt upon intereſt; the annuity itſelf being, properly, the ſimple intereſt, and the capital, from whence it ariſes, the principal. It is true, the ſum $1 + nr$, expreſſing the amount of 1 *l.* is given, ſtrictly ſpeaking, according to *ſimple intereſt*: but the concluſion (as a late author * very juſtly obſerves) would be more congruous, and anſwer better, were the ſame allowances to be made therein, as are made in finding the amount of the annuity; that is, were intereſt upon intereſt to be taken once and no more. Agreeable to this aſſumption, r, the intereſt of 1 *l.* being conſidered as an annuity, its amount in n years (by writing r for A, in the general *formula* above) will be given $= nr + \frac{1}{2}n \,.\, \overline{n-1} \,.\, r^2$: to which the principal 1 *l.* being added, the aggregate $1 + n + \frac{1}{2}n \,.\, \overline{n-1} \,.\, r^2$ will therefore be the whole amount of 1 *l.* in the time n; and ſo we ſhall have $1 + nr + \frac{1}{2}n \,.\, \overline{n-1} \,.\, r^2 : 1 :: nA + \frac{1}{2}n \,.\, \overline{n-1} \,.\, rA : \dfrac{2nA + n \,.\, \overline{n-1} \,.\, rA}{2 + 2r + n \,.\, \overline{n-1} \,.\, r^2} = v$, the true value of the annuity, *according to the ſaid hypotheſis*. From which equation others may be derived, by means whereof the different values of A, n, and r may be, ſucceſſively, determined. But, as this method of allowing intereſt upon intereſt, once and no more, is arbitrary, and the valuation of annuities, according to ſimple intereſt, a matter of more ſpeculation than real uſe, it being not only cuſtomary, but alſo moſt equitable to allow compound intereſt in theſe caſes, I ſhall not ſtay to exemplify *it*, but proceed to

The Reſolution of the various caſes of compound intereſt, and of Annuities, as depending thereon.

Let $\begin{cases} R = \begin{cases}\text{the amount of 1 } l. \text{ in one year, } viz. \text{ principal and intereſt.}\end{cases} \\ P = \text{any ſum put out at intereſt.}\end{cases}$

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* *Mr. Hardy in his Annuities.*

Let
n = the number of years it is lent for.
a = its amount in that time.
A = any annuity forborn n years.
m = its amount.
v = the preſent value of the annuity for the ſame time.

Therefore, ſince one pound, put out at intereſt, in the firſt year is increaſed to R, it will be as 1 to R, ſo is R, the ſum forborn the ſecond year, to R^2, the amount of one pound in two years; and therefore as 1 to R, ſo is R^2, the ſum forborn the third year, to R^3, the amount in three years: whence it appears that R^n, or R raiſed to the power whoſe exponent is the number of years, will be the amount of one pound in thoſe years. But as 1 *l.* is to its amount R^n, ſo is P to (a) its amount, in the ſame time; whence we have $P \times R^n = a$. Moreover, becauſe the amount of one pound, in n years, is R^n, its increaſe in that time will be $R^n - 1$; but its intereſt for one ſingle year, or the annuity anſwering to that increaſe, is $R - 1$; therefore as $R - 1$ to $R^n - 1$, ſo is A to m. Hence we get $\frac{A \times \overline{R^n - 1}}{R - 1} = m$. Furthermore, ſince it appears that one pound, ready-money, is equivalent to R^n, to be received at the expiration of n years, we have, as R^n to 1, ſo is $\frac{A \times \overline{R^n - 1}}{R - 1}$ (the ſum in arrear) to v, its worth in ready-money; and therefore $\frac{A \times \overline{1 - \frac{1}{R^n}}}{R - 1} = v$.

From which three original equations, others may be derived, by help whereof the various queſtions relating to compound intereſt, annuities in arrear, and the preſent values of annuities, may be reſolved.

Thus, becauſe PR^n is $= a$, there will come out $P = \frac{a}{R^n}$, and $R = \left.\frac{a}{P}\right|^{\frac{1}{n}}$, &c. or, by exhibiting the ſame equations in logarithms (which is the moſt eaſy for practice) we ſhall have

1° $\text{Log. } a = \text{log. } P + n \times \text{log. } R.$
2° $\text{Log. } P = \text{log. } a - n \times \text{log. } R.$
3° $\text{Log. } R = \frac{\text{log. } a - \text{log. } P}{n}.$
4° $n = \frac{\text{log. } a - \text{log. } P}{\text{log. } R}.$

Which four theorems, or equations, ſerve for the four caſes in compound intereſt.

Again, ſince m is $= \frac{A \times \overline{R^n - 1}}{R - 1}$, we ſhall have

1°. $\text{Log. } m = \text{log. } A + \text{log. } \overline{R^n - 1} - \text{log. } \overline{R - 1}.$
2°. $\text{Log. } A = \text{log. } m - \text{log. } \overline{R^n - 1} + \text{log. } \overline{R - 1}.$
3°. $n = \frac{\text{log. } \overline{mR - m + A} - \text{log. } A}{\text{log. } R}.$
4°. $R^n - \frac{mR}{A} + \frac{m}{A} - 1 = 0.$

To which the various queſtions relating to annuities in arrear are referred.

Moreover, ſeeing $A \times \frac{1 - \frac{1}{R^n}}{R - 1}$ is $= v$, we thence have,

1°. $\text{Log. } v = \text{log. } A + \text{log. } \overline{1 - \frac{1}{R^n}} - \text{log. } \overline{R - 1}.$
2°. $\text{Log. } A = \text{log. } v + \text{log. } \overline{R - 1} - \text{log. } \overline{1 - \frac{1}{R^n}}.$
3°. $n = \frac{\text{log. } A - \text{log. } \overline{A + v - vR}}{\text{log. } R}.$
4°. $R^{n+1} - \overline{\frac{A}{v} + 1} \times R^n + \frac{A}{v} = 0.$

The uſe of which theorems, reſpecting the preſent values of annuities, as well as of the preceding ones, for compound intereſt and annuities in arrear, will fully appear from the following examples.

Examp. 1. To find the amount of 575 *l.* in ſeven years, at four *per cent. per annum*, compound intereſt.

In this caſe we have given $P = 575$, $R = 1{,}04$, and $n = 7$; therefore, *by Theorem* 1, $\log. a = \log. 575 + 7 \log. 1{,}04 = 2{,}8789011$; and conſequently $a = 756{,}66$, or 756 *l.* 13 *s.* $2\frac{1}{4}$ *d.* the value required.

Examp. 2. What principal, put to intereſt, will raiſe a ſtock of 1000 *l.* in fifteen years, at 5 *per cent.*?

Here, we have given $R = 1{,}05$, $n = 15$, and $a = 1000$; therefore, *by Theorem* 2, $\log. P = \log. 1000 - 15 \log. 1{,}05 = 2{,}6821605$; and conſequently $P = 481{,}02$, or 481 *l.* 0 *s.* $4\frac{3}{4}$ *d.*, the value ſought.

Examp. 3. In how long time will 575 *l.* raiſe a ſtock of 756 *l.* 13 *s.* $2\frac{1}{4}$ *d.* at 4 *per cent.*?

In this caſe we have $R = 1{,}04$, $P = 575$, and $a = 756.66$; whence, *by Theor.* 4, $n = \dfrac{\log. 756{,}66 - \log. 575}{\log. 1{,}04} = 7$, the number of years required.

Examp. 4. To find at what rate of intereſt 481 *l.* in fifteen years, will raiſe a ſtock of 1000 *l.*

Here we have given $P = 481$, $a = 1000$, and $n = 15$; therefore, *by Theorem* 3, $\log. R = \dfrac{\log. 1000 - \log. 481}{15} = .0211903$, whence $R = 1{,}05$; conſequently 5 *per cent.* is the rate required.

The four laſt examples relate to the caſes in compound intereſt; the four next are upon the forbearance of annuities.

Examp. 1. If 50 *l.* yearly rent, or annuity, be forborn ſeven years, what will it amount to, at 4 *per cent. per annum*, compound intereſt?

Here, we have $R = 1{,}04$, $A = 50$, and $n = 7$; and therefore, *by Theor.* 1, log. m (= log. A + log. $\overline{R^n - 1}$ — log. $\overline{R - 1}$) = log. 50 + log. $\overline{1{,}04|^7 - 1}$, — log. ,04 = 2,596597; and conſequently m = 395*l.* the value that was to be found.

Examp. 2. What annuity, forborn ſeven years, will amount to, or raiſe a ſtock of 395*l.* at 4 *per cent.* compound intereſt?

In this caſe we have given R = 1,04, n = 7, and m = 395; whence, *by Theorem* 2, log. A (= log. m — log. $\overline{R^n - 1}$ + log. $\overline{R - 1}$) = log. 395 — log. $\overline{1{,}04|^7 - 1}$ + log. ,04 = 1,6989700; and conſequently A = 50*l.* which is the annuity required.

Examp. 3. In how long time will 50*l.* annuity raiſe a ſtock of 395*l.* at 4 *per cent. per annum*, compound intereſt?

Here, we have R = 1,04, A = 50, m = 395; and therefore, *by Theor.* 3, n $\left(= \frac{\text{log.}\,\overline{mR - m + A} - \text{log.}\,A}{\text{log.}\,R}\right)$ $= \frac{,1192559}{,0170333} = 7$, the number of years required.

Examp. 4. If 120*l.* annuity, forborn eight years, amounts to, or raiſes a ſtock of 1200*l.* what is the rate of intereſt?

In this caſe we have given n = 8, A = 120, and m = 1200, to find R, therefore, *by Theorem* 4, we have $R^8 - 10R + 9 = 0$; from which, by any of the methods in Sect. 13, the required value of R will be found = 1,c6287; therefore the rate is 6,287, or 6*l.* 5*s.* 9*d.* *per cent. per annum.*

The ſolution of the laſt caſe, where the rate is required, being a little troubleſome, I ſhall here put down an approximation (derived from the third general *formula*, at *p.* 165) which will be found to anſwer very near the truth, provided the number of years is not very great.

Let $Q = \frac{n . \overline{n-1} . A}{2 . \overline{m - nA}}$; then will

$$\frac{3000 Q + \overline{2n-1} . 400}{6Q . \overline{5Q + 3n - 4} + \frac{1}{6} . \overline{n-2} . \overline{11n - 13}}$$ be the rate *per cent. required.*

Thus, for example, let $n = 8$, $A = 120$, and $m = 1200$; then will $Q = \frac{56 . 120}{2 . 240} = 14$, and the rate itſelf $= \frac{42000 + 6000}{84 \times 90 + 75} = 6{,}287$, *as above.*

The preceding examples explain the different caſes of *annuities in arrear*; in the following ones the rules for the *valuation of annuities* are illuſtrated.

Examp. 1. To find the preſent value of 100*l.* annuity, to continue ſeven years, allowing 4 *per cent. per annum* compound intereſt.

Here, we have given $R = 1{,}04$, $A = 100$, and $n = 7$; and therefore, *by Theorem* 1, log. v (= log. A + log. $\overline{1 - \frac{1}{R^n}}$ — log. $\overline{R - 1}$) = log. 100 + log. $\overline{1 - \frac{1}{\overline{1{,}04}|^7}}$ — log. ,04 = 2,778296; and conſequently $v = 600{,}2 = 600$*l.* 4*s.* which is the value that was to be found.

Examp. 2. What annuity, or yearly income, to continue 20 years, may be purchaſed for 1000*l.* at 3½ *per cent.*?

In this caſe, $R = 1{,}035$, $n = 20$, $v = 1000$; whence, *by Theorem* 2, we have log. A (= log. v + log. $\overline{R - 1}$ — log. $\overline{1 - \frac{1}{R^n}}$) = 1,847336; and conſequently A = 70,36, or 70*l.* 7*s.* 2*d.*

Examp. 3. For how long time may one, with 600*l.* purchaſe an annuity of 100*l.* at 4 *per cent.*?

In this example we have $R = 1{,}04$, $A = 100$, and $v = 600$; and therefore, *by Theorem* 3, $n \left(= \frac{\log. A - \log. \overline{A + v - vR}}{\log. R} \right) = 7$, the number of years required.

Examp. 4. To determine at what rate of intereſt, an annuity of 50*l.* to continue 10 years, may be purchaſed, for 400*l.*

Here $A = 50$, $n = 10$, and $v = 400$; whence, *by Theorem* 4, $R^{n+1} - \overline{\frac{A}{v} + 1} \times R^n + \frac{A}{v}$ being $= 0$, we have $R^{11} - 1{,}125R^{10} + {,}125 = 0$; which equation reſolved, gives the required value of $R = 1{,}042775$; and conſequently the rate of intereſt, 4,2775*l. per annum.*

The ſolution of this laſt caſe being ſomewhat tedious, the following approximation (which will be found to anſwer very near the truth when the number of years is not very large) may be of uſe.

Aſſume $Q = \frac{n \,.\, \overline{n + 1} \,.\, A}{2nA - 2v}$; ſo ſhall

$$\frac{3000Q - \overline{2n + 1} \times 400}{6Q \,.\, \overline{5Q - 3n - 4} + \frac{1}{6} \,.\, \overline{n + 2} \,.\, \overline{11n + 13}}$$

expreſs the rate *per cent.* very nearly.

Thus, for example, let A (as above) be $= 50$, $n = 10$, and $v = 400$; then, Q being $= \frac{10 \times 11 \times 50}{1000 - 800} = 27{,}5$, we have $\frac{82500 - 8400}{165 \times 103{,}5 + 246}$, or 4,2775, for the rate, *per cent.* the ſame as before.

SECTION XVII.

Of Plane Trigonometry.

DEFINITIONS.

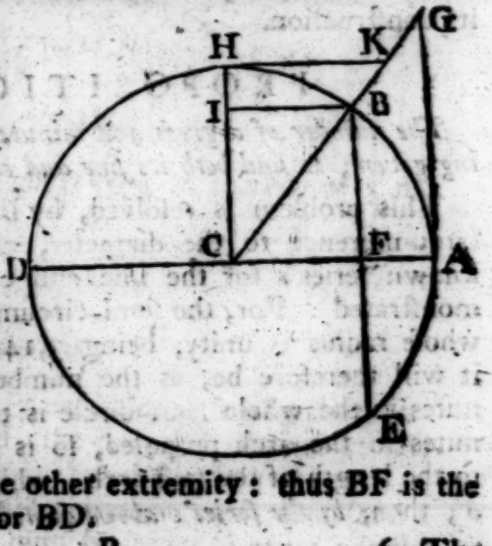
1. PLANE Trigonometry is the art whereby, having given any three parts of a plane triangle (except the three angles) the reſt are determined. In order to which, it is not only requiſite that the peripheries of circles, but alſo that certain right lines, in and about the circle, be ſuppoſed divided into ſome aſſigned number of equal parts.

2. The periphery of every circle is ſuppoſed to be divided into 360 equal parts, called degrees; and each degree into 60 equal parts, called minutes; and each minute into 60 equal parts, called ſeconds, or ſecond minutes, *&c.* Any part of the periphery is called an arch, and is meaſured by the number of degrees and minutes, *&c.* it contains.

3. The difference of any arch from 90 degrees, or a quadrant, is called its complement, and its difference from 180 degrees, or a ſemicircle, its ſupplement.

4. A chord, or ſubtenſe, is a right line drawn from one extremity of an arch to the other; thus BE is the chord or ſubtenſe of the arch BAE, or BDE.

5. The ſine (or right ſine) of an arch, is a right line drawn from one extremity of the arch perpendicular to the diameter paſſing through the other extremity: thus BF is the ſine of the arch AB, or BD.



6. The verſed ſine of an arch is the part of the diameter intercepted between the arch and its ſine: ſo AF is the verſed ſine of AB, and DF of DB.

7. The co-ſine of an arch is the part of the diameter intercepted between the center and the ſine; and is equal to the ſine of the complement of that arch. Thus CF is the co-ſine of the arch AB, and is equal to BI, the ſine of its complement HB.

8. The tangent of an arch, is a right line touching the circle in one extremity of that arch, continued from thence to meet a line drawn from the center through the other extremity; which line is called the ſecant of the ſame arch: thus AG is the tangent, and CG the ſecant of the arch AB.

9. The co-tangent and co-ſecant of an arch, are the tangent and ſecant of the complement of that arch: thus HK and CK are the co-tangent and co-ſecant of the arch AB.

10. A trigonometrical canon, is a table exhibiting the length of the ſine, tangent, &c. to every degree and minute of the quadrant, with reſpect to the radius, which is ſuppoſed unity, and conceived to be divided into 10000000 or more decimal parts. Upon this table the numerical ſolution of the ſeveral caſes in trigonometry depend; it will therefore be proper to begin with its conſtruction.

PROPOSITION I.

The number of degrees and minutes, &c. in an arch being given; to find both its ſine and co-ſine.

This problem is reſolved, by having the ratio of the circumference to the diameter, and by means of the known ſeries's for the ſine and co-ſine (hereafter demonſtrated). For, the ſemi-circumference of the circle, whoſe radius is unity, being 3,14159265358979 &c. it will therefore be, as the number of degrees or minutes in the whole ſemi-circle is to the degrees or minutes in the arch propoſed, ſo is 3,14159265358 &c. to the length of the ſaid arch; which let be denoted by a; then, *by the ſeries's above quoted*, its ſine will be expreſſed by $a - \frac{a^3}{2.3} + \frac{a^5}{2.3.4.5} - \frac{a^7}{2.3.4.5.6.7}$ *&c.* and its co-ſine by $1 - \frac{a^2}{2} + \frac{a^4}{2.3.4} - \frac{a^6}{2.3.4.5.6} + \frac{a^8}{2.3.4.5.6.7.8}$ *&c.*

Thus, for example, let it be required to find the ſine of one minute: then, as 10800 (the minutes in 180 degrees): 1 : : 3,14159265358 *&c.* : .000290888208665 = the length of an arch of one minute: therefore, in this caſe, $a = .000290888208665$, and $\frac{a^3}{2.3} \left(= \frac{a^3}{6}\right) = .000000000004102$ *&c.* And conſequently .000290888204563 = the required ſine of one minute.

Again, let it be required to find the ſine and co-ſine of five degrees, each true to ſeven places of decimals. Here ,0002908882, the length of an arch of 1 minute (found above) being multiplied by 300, the number of minutes in 5 degrees, the product .08726646 will be the length of an arch of 5 degrees: therefore, in this caſe, we have

$$\begin{aligned} a &= ,08726646, \\ -\frac{a^3}{6} &= -,00011076, \\ +\frac{a^5}{120} &= +,00000004, \end{aligned}$$

&c. and conſequently ,08715574 = the ſine of 5 degrees. Alſo

$$\begin{aligned} \frac{a^2}{2} &= ,00380771, \\ \frac{a^4}{24} &= ,00000241\,; \end{aligned}$$

and conſequently ,9961947 = the co-ſine of 5 degrees.

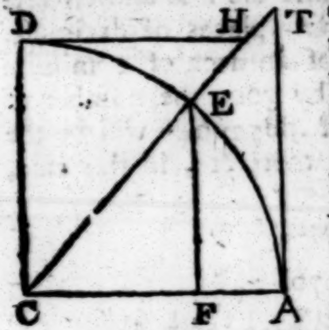
After the ſame manner, the ſine and co-ſine of any other arch, may be derived; but the greater the arch is, the ſlower the ſeries will converge, and therefore a greater number of terms muſt be taken to bring out the concluſion to the ſame degree of exactneſs.

But there is another method of conſtructing the trigonometrical canon; which, though leſs direct, is more geometrical; and that is, by determining the ſines and tangents of different arches, one from another, as in the enſuing propoſitions.

PROPOSITION II.

The ſine of an arch being given; to find its co-ſine, tangent, co-tangent, ſecant, and co-ſecant.

Let AE be the propoſed arch, EF its ſine, CF its co-ſine, AT its tangent, DH its co-tangent, CT its ſecant, and CH its co-ſecant: then, *by Euc.* 47. 1. we ſhall have CF $= \sqrt{CE^2 - EF^2}$; from whence the co-ſine will be known; and then, by reaſon of the ſimilar triangles, CFE, CAT, and CDH, it will be,



1. CF : FE : : CA : AT; whence the tangent is known.

2. CF : CE : : CA : CT; whence the ſecant is known.

3. EF : CF : : CD : DH; whence the co-tangent is known.

4. EF : CE : : CD : CH; whence the co-ſecant is alſo known.

Hence it appears,

1. That the tangent is a fourth proportional to the co-ſine, the ſine, and the radius.

2. That the ſecant is a third proportional to the co-ſine, and the radius.

3. That the co-tangent is a fourth proportional to the ſine, the co-ſine, and the radius.

4. That the co-ſecant is a third proportional to the ſine, and the radius.

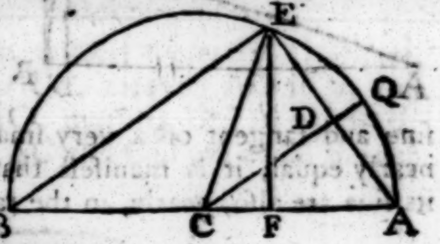
5. And that the rectangle of the tangent and co-tangent is equal to the ſquare of the radius.

PROPOSI-

PROPOSITION III.

The co-ſine CF *of an arch* AE, *being given; to find the ſine and co-ſine of half that arch.*

From the two extremities of the diameter AB draw the ſubtenſes AE and BE; and let CQ biſect the arch AE in Q, and its chord (perpendicularly) in D; then, ſince the angle BEA is a right one (*by Euc.* 31. 3.) the triangles ABE and ADC are ſimilar; and therefore, AC being $= \frac{1}{2}$ AB, AD muſt be $= \frac{1}{2}$ AE, and CD $= \frac{1}{2}$ BE: but AE is $= \sqrt{AB \times AF}$, and BE $= \sqrt{AB \times BF}$; therefore



$$\left.\begin{array}{l} AD=\tfrac{1}{2}\sqrt{AB\times AF}=\sqrt{\tfrac{1}{2}AC\times AF}=\text{the ſine}\\ CD=\tfrac{1}{2}\sqrt{AB\times BF}=\sqrt{\tfrac{1}{2}AC\times BF}=\text{the co-ſine}\end{array}\right\}\text{of }\tfrac{1}{2}\,AE.$$

Hence it is evident, that the ſine of the half of any arch, is a mean proportional between half the radius, and the verſed-ſine of the whole arch; and its co-ſine, a mean proportional between half the radius and the verſed-ſine of the ſupplement of the ſame arch.

PROPOSITION IV.

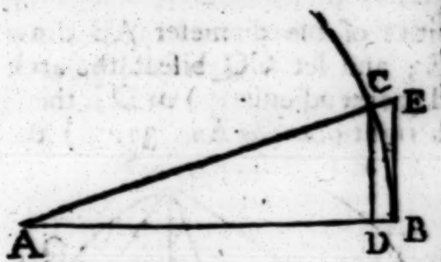
The ſine AD, *and co-ſine* CD, *of an arch* AQ *being given; to find* EF *the ſine of the double of that arch (ſee the preceding figure.)*

Since AE = 2AD and BE = 2CD, and the triangles ABE and AEF are alike (*by Euc.* 8. 6.) we have, as AB (2AC) : AE (2AD) :: BE (2CD) : EF; whence it appears, that the ſine of double any arch is a fourth proportional to the radius, the ſine, and the double co-ſine of the ſame arch.

PROPOSITION V.

The ſine CD *and tangent* BE, *of a very ſmall arch are, nearly, in the ratio of equality.*

For, the triangles ADC and ABE being ſimilar, thence will AD : AB : : DC : BE : but as the point C approaches to B, the difference of AB and AD will become indefinitely ſmall in reſpect of AB, and therefore the difference of BE and DC will likewiſe become indefinitely ſmall with reſpect to BE or DC.



Corollary. Becauſe any arch BC is greater than its ſine and leſs than its tangent; and ſince the ſine and tangent of a very ſmall arch are proved to be nearly equal, it is manifeſt that a very ſmall arch and its ſine are alſo, nearly, in the ratio of equality.

PROPOSITION VI.

To find the ſine of an arch of one minute.

The ſine of 30 degrees is known, being half the chord of 60 degrees, or the radius; therefore, *by Prop.* 2. *and* 3, the ſine of 15 degrees will be known: and, the ſine of 15 degrees being known, the ſine of 7° 30′ will be found (*by the ſame Propoſitions*), and from thence the ſine of 3° 45′; and ſo likewiſe the ſine of half *this*; and ſo on, till 12 biſections being made, we come, at laſt, to the ſine of an arch of $52''$, $44'''$, $03''''$, $45'''''$; which ſine (*by Corol. to the preceding Prop.*) will (as the co-ſine is nearly equal to the radius) be nearly equal to the arch itſelf. Therefore we have, as $52''$, $44'''$, $03''''$, $45'''''$, is to 1′, ſo is the length of the former of theſe arches (found as above) to the length of an arch of one minute, or that of its ſine, very nearly.

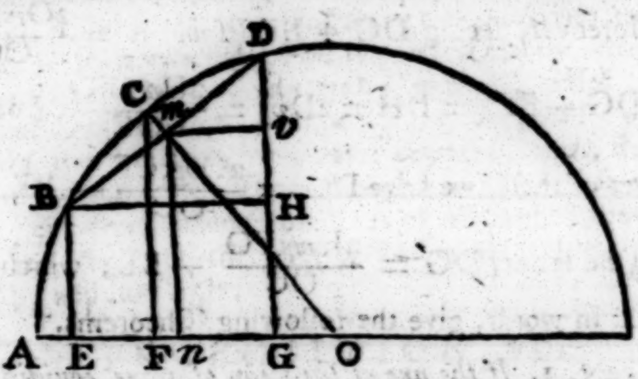
If it be taken for granted, that 3,1415926535 &c. is the length of half the periphery of the circle whoſe radius is unity, we ſhall have, as 10800, the number of minutes in 180°, or the whole ſemi-circle, is to one minute, ſo is 3,1415926535 &c. the whole ſemi-circle, to 0,000290888208, the length of an arch of one minute, or *that* of its ſine, very nearly.

PROPO-

PROPOSITION VII.

If there be three equidifferent arches AB, AC, *and* AD, *it will be, as the radius is to the co-ſine of their common difference* BC *or* CD, *ſo is the ſine,* CF, *of the mean, to half the ſum of the ſines,* BE + DG, *of the two extremes: and as the radius is to the ſine of the common difference, ſo is the co-ſine* FO *of the mean, to half the difference of the ſines of the two extremes.*

For, let BD be drawn, cutting the radius OC in *m*, alſo draw *mn* parallel to CF, meeting AO in *n*, and BH and *mv* parallel to AO, meeting DG in H and *v*: then, becauſe the arches BC and CD are equal to each other, OC is not only perpendicular to BD, but alſo biſects it (*Euc.* 3. 3.) whence it is evident that B*m*, or D*m*, will be the ſine of BC, or CD, and O*m* its co-ſine; and that *mn*, being an arithmetical mean between the ſines, BE and DG, of the two extremes, is equal to half their ſum, and D*v* equal to half their difference. Moreover, by reaſon of the ſimilarity of the triangles OCF, O*mn*, and D*mv*, it will



$$\left.\begin{array}{l}\text{be as, } OC : Om :: CF : mn \\ \text{and as, } OC : Dm :: FO : Dv\end{array}\right\}\text{Q. E. D.}$$

COROL. 1.

Since, from the foregoing proportions, *mn* is $= \frac{Om \times CF}{OC}$, and D*v* (= *v*H) $= \frac{Dm \times FO}{OC}$, it is evident that DG (= *mn* + D*v*) will be = $\frac{Om\times CF + Dm\times FO}{OC}$. and BE (= *mn*—*v*H) = $\frac{Om\times CF - Dm\times FO}{OC}$: from whence it appears, that the ſine (DG) of the ſum (AD) of any two arches (AC and CD) is equal to the ſum of the rectangles of the ſine of the one into the co-ſine of the other, alternately, divided by the radius; and that the ſine (BE) of their difference (AB) is equal to the difference of the ſame rectangles, divided alſo by the radius.

COROL. 2.

Moreover, ſeeing DG + BE (2*mn*) is = $\frac{2Om\times CF}{OC}$, and DG — BE (= DH = 2D*v*) = $\frac{2Dm\times FO}{OC}$, from the former of theſe, we have DG = $\frac{2Om\times CF}{OC}$ — BE, and, from the latter, DG = $\frac{2Dm\times FO}{OC}$ + BE; which, expreſſed in words, give the following Theorems.

Theor. 1. *If the ſine of the mean of three equidifferent arches (ſuppoſing the radius unity) be multiplied by twice the co-ſine of the common difference, and the ſine of either extreme be ſubtracted from the product, the remainder will be the ſine of the other extreme.*

Theor. 2. *Or, if the co-ſine of the mean be multiplied by twice the ſine of the common difference, and the product be added to, or ſubtracted from the ſine of one of the extremes, the ſum or remainder will be the ſine of the other extreme.*

Theſe two theorems are of excellent uſe in the conſtruction of the trigonometrical-canon: for, ſuppoſing the ſine and co-ſine of an arch of 1 minute to be found, by Prop. 6 and 1, and to be denoted by p and q, reſpectively; then, the ſine of 2 minutes being given from Prop. 4, the ſine of 3 minutes will from hence be known, being $= 2q\times$ ſine $2'$ — ſine $1'$ (*by Theor.* 1) or $= 2p$ $\times$ co-ſine of $2'$ + ſine of $1'$ (*by Theor.* 2.) After the ſame manner the ſine of $4'$ will be found, being $= 2q \times$ ſine of $3'$ — ſine of $2'$; or $= 2p \times$ co-ſine of $3'$ + ſine of $2'$. And thus the ſines of 5, 6, 7, &c. minutes may be ſucceſſively derived, by either of the theorems; but the former is the moſt commodious.

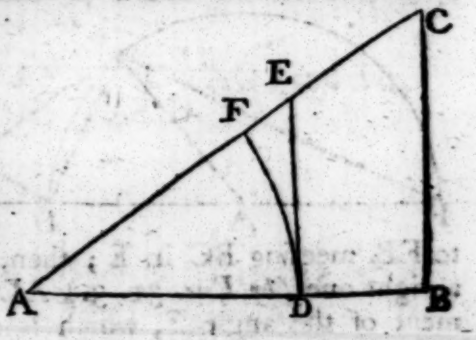
If the mean arch be 45°; then its co-ſine being $= \sqrt{\frac{1}{2}}$, it follows (*from Theorem* 2.) that the ſine of the exceſs of any arch above 45°, multiplied by $2\sqrt{\frac{1}{2}}$ or $\sqrt{2}$, gives the exceſs of the ſine of that arch above *that* of another arch as much below 45°; thus $\sqrt{2} \times$ ſine of 10° = ſine of 55° — ſine of 35°; and $\sqrt{2} \times$ ſine of 15° = ſine of 60° — ſine of 30°; and ſo of others: which is uſeful in finding the ſines of arches greater than 45°.

But, if the mean arch be 60 degrees, then its co-ſine being $\frac{1}{2}$, it is evident, *from the ſame Theorem*, that the ſine of the exceſs of any arch above 60°, added to the ſine of another arch as much below 60°, will give the ſine of the firſt arch, or greater extreme: thus, the ſine of 10° + ſine 50° = ſine 70°, and ſine 15° + ſine 45° = ſine 75°; from whence the ſines of all arches above 60 degrees, thoſe of the inferior arches being known, are had by addition only.

PROPOSITION VIII.

In any right-angled plane triangle ABC, *it will be, as the baſe* AB *is to the perpendicular* BC, *ſo is the radius (of the tables) to the tangent of the angle at the baſe.*

Let DA be the radius to which the table of ſines and tangents is adapted, and DE the tangent of the angle A; then, by reaſon of the ſimilarity of the triangles ABC and ADE, it will be, as AB : BC : : AD : DE. Q.E.D.

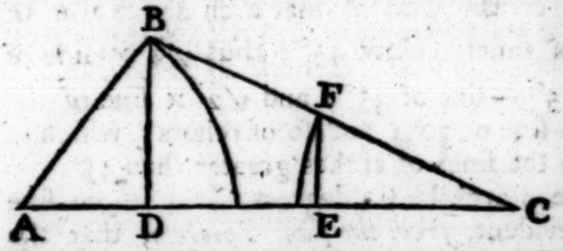


PROPO-

PROPOSITION IX.

In every plane triangle it will be, as any one ſide is to the ſine of the oppoſite angle; ſo is any other ſide to the ſine of its oppoſite angle.

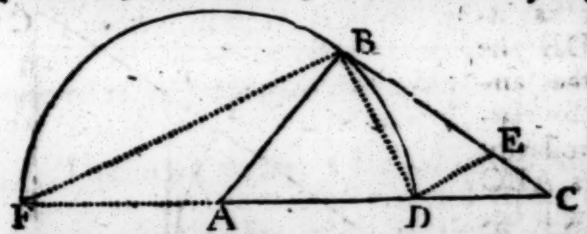
For, let ABC be the propoſed triangle; take CF = AB, and upon AC, let fall the perpendiculars BD and EF; which will be the ſines of the angles A and C, to the equal radii AB and CF. But the triangles CBD and CFE are ſimilar, and therefore CB : BD :: CF (AB) : FE; that is, as CB is to the ſine of A, ſo is AB to the ſine of C. Q. E. D.



PROPOSITION X.

In every plane triangle, it will be, as the ſum of any two ſides is to their difference, ſo is the tangent of the complement of half the angle included by thoſe ſides, to the tangent of the difference of either of the other two angles and the ſaid complement.

For, let ABC be the triangle, and AB and AC the two propoſed ſides; and upon A, as a center, with the radius AB, let a ſemicircle be deſcribed, cutting CA, produced, in D and F; ſo that CF may expreſs the ſum, and CD the difference of the ſides AC and AB; join F, B and B, D, and draw DE parallel to FB, meeting BC in E; then, the angle FBD being a right one (*by Euc.* 31. 3.) ADB will be the complement of the angle F, which is equal to half the propoſed angle A, *by Euc.* 20. 3. Moreover, ſeeing the angles

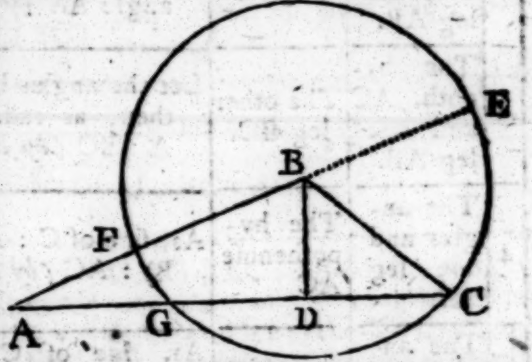


angles FBD and EDB are both right-ones; (for EDB is = FBD (= right angle) becauſe DE is parallel to FB) it is plain, that, if BD be made the radius, BF will be the tangent of BDF, and DE the tangent of DBE: but, becauſe of the ſimilar triangles CFB and CDE, CF : CD : : BF : DE; that is, as the ſum of the ſides AC and AB, is to their difference; ſo is the tangent of BDF, to the tangent of DBC; which angle is, manifeſtly, the exceſs of ABC, above BDF, or ABD; and alſo the exceſs of ADB above ACB. *Q.E.D.*

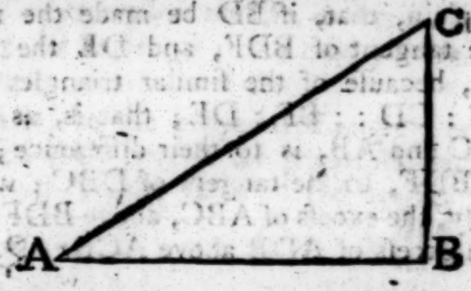
PROPOSITION XI.

As the baſe of any plane triangle is to the ſum of the two ſides, ſo is the difference of the ſides to the difference of the ſegments of the baſe, made by a perpendicular falling from the vertical angle.

For, let ABC be the propoſed triangle, and BD the perpendicular; from B as a center, with the interval BC, let the circumference of a circle be deſcribed, cutting the baſe AC in G and the ſide AB, produced, in F and E: then will AE be the ſum of the ſides, AF their difference, and AG the difference of the ſegments of the baſe AD and DC: but (*by Euc.* 36. 3.) AE × AF = AC × AG; and therefore AC : AE : : AF : AG. Q.E.D.

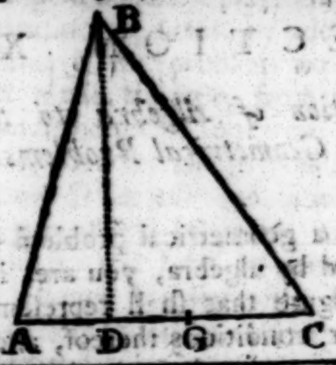


The Solution of the caſes of right-angled plane triangles.



Caſe	Given.	Sought.	Proportion.
1	The hypothenuſe AC and the angles	The leg BC.	As the radius (or the ſine of B) is the hyp. AC; ſo is the ſine of A, to its oppoſite ſide BC *(by Prop.* 9.)
2	The hypoth. AC and one leg AB.	The angles.	As, AC : rad. : : AB : ſine of C; whoſe complement gives the angle A.
3	The hypoth. AC and one leg AB.	The other leg BC.	Let the angles be found by caſe 2; then, as rad. : AC : : ſine of A : BC *(by Prop.* 9.)
4	The angles and one leg AB.	The hypothenuſe AC.	As, ſine of C : AB : : rad. (ſine of B) : AC *(by Prop.* 9.)
5	The angles and one leg AB.	The other leg BC.	As, ſine of C : AB : : ſine of A : BC *(by Prop.* 9.) Or, rad. : tang. of A : : AB : BC *(by Prop.* 8.)
6	The two legs AB and BC.	The angles.	As, AB : BC : : rad. : tang, of A, *(by Prop.* 8); whoſe complement gives the angle C.
7	The two legs AB and BC.	The hypothenuſe AC.	Find the angles, by caſe 6, and from thence the hyp. AC, by caſe 4.

The Solution of the caſes of oblique plane triangles.



Caſe	Given.	Sought.	Proportion.
1	The angles and 1 ſide AB.	Either of the other ſides, ſuppoſe BC.	As, ſine of C : AB :: ſine of A : BC (*by Prop.* 9.)
2	Two ſides AB, BC and the angle C oppoſite to one of them.	The other angles A and ABC.	As, AB : ſine of C :: BC : ſine of A; which added to C, and the ſum ſubtracted from 180°, gives the angle ABC.
3	Two ſides AB, BC and the angle C oppoſite to one of them.	The other ſide AC.	Find the angle ABC, by caſe 2; then, as ſine of A : BC :: ſine of ABC : AC.
4	Two ſides AB, AC and the included angle A.	The other angles C and ABC.	As, AB + AC : AB — AC :: tang. of the comp. of $\frac{1}{2}$ A : tang. of an ang. which added to the ſaid com. gives the greater ang. C; and ſubtracted leaves the leſſer ABC (*Prop.* 10.)
5	Two ſides AB, AC and the included angle A.	The other ſide BC.	Find the angles, by caſe 4; and then BC, by caſe 1.
6	All the ſides.	An angle, ſuppoſe A.	Let fall a perpendicular BD, oppoſite the required angle, and ſuppoſe DG = AD; then (*by Prop.* 11,) AC : BC + BA :: BC — BA : CG, which ſubtracted from AC, and the remainder divided by 2 gives AD; whence A will be found, by caſe 2, of right angles.

SEC-

SECTION XVIII.

The Application of Algebra to the Solution of Geometrical Problems.

WHEN a geometrical problem is propoſed to be reſolved by algebra, you are, in the firſt place, to deſcribe a figure that ſhall repreſent, or exhibit the ſeveral parts or conditions thereof, and look upon that figure as the true one; then, having conſidered attentively the nature of the problem, you are next to prepare the figure for a ſolution (if need be) by producing, and drawing, ſuch lines therein, as appear moſt conducive to that end. This done, let the unknown line, or lines which you think will be the eaſieſt found (whether required or not) together with the known ones (or as many of them as are requiſite) be denoted by proper ſymbols; then proceed to the operation, by obſerving the relation that the ſeveral parts of the figure have to each other; in order to which, a competent knowledge in the elements of geometry is abſolutely neceſſary.

As no general rule can be given for the drawing of lines, and electing the moſt proper quantities to ſubſtitute for, ſo as to, always, bring out the moſt ſimple concluſions (becauſe different problems require different methods of ſolution); the beſt way, therefore, to gain experience in this matter, is to attempt the ſolution of the ſame problem ſeveral ways, and then apply that which ſucceeds beſt, to other caſes of the ſame kind, when they afterwards occur. I ſhall, however, ſubjoin a few general directions, which will be found of uſe.

1°. In preparing the figure, by drawing lines, let them be either parallel or perpendicular to other lines in the figure, or ſo as to form ſimilar triangles; and, if an angle be given, let the perpendicular be oppoſite to that angle, and alſo fall from the end of a given line, if poſſible.

2°. In electing proper quantities to ſubſtitute for, let thoſe be choſen (whether required or not) which lie neareſt the known, or given parts of the figure, and by help whereof the next adjacent parts may be expreſſed, without the intervention of ſurds, by addition and ſubtraction only. Thus, if the problem were to find the perpendicular of a plane triangle, from the three ſides given, it will be much better to ſubſtitute for one of the ſegments of the baſe, than for the perpendicular, though the quantity required; becauſe the whole baſe being given, the other ſegment will be given, or expreſſed, by ſubtraction only, and ſo the final equation come out a ſimple one; from whence the ſegments being known, the perpendicular is eaſily found by common arithmetick: whereas, if the perpendicular were to be firſt ſought, both the ſegments would be ſurd quantities, and the final equation an ugly quadratic one.

3°. When, in any problem, there are two lines or quantities alike related to other parts of the figure, or problem, the beſt way is to make uſe of neither of them, but to ſubſtitute for their ſum, their rectangle, or the ſum of their alternate quotients, or for ſome line or lines, in the figure, to which they have both the ſame relation. This rule is exemplified in Prob. 22, 23, 24, and 27.

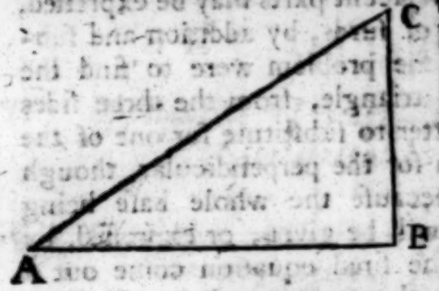
4°. If the area, or the perimeter of a figure be given, or ſuch parts thereof as have but a remote relation to the parts required, it will, ſometimes, be of uſe to aſſume another figure ſimilar to the propoſed one, whereof one ſide is unity, or ſome other known quantity; from whence the other parts of this figure, by the known proportions of the homologous ſides, or parts, may be found, and an equation obtained, as is exemplified in Prob. 25 and 32.

Theſe are the moſt general obſervations I have been able to collect; which I ſhall now proceed to illuſtrate by proper examples.

PROBLEM I.

The baſe (b), *and the ſum of the hypothenuſe and perpendicular* (a) *of a right-angled triangle* ABC, *being given; to find the perpendicular.*

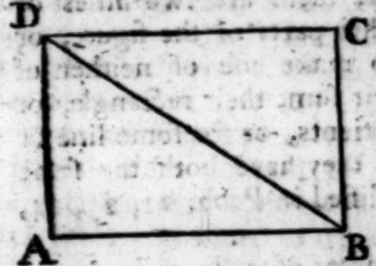
Let



Let the perpendicular BC be denoted by x; then the hypothenuſe AC will be expreſſed by $a - x$: but (*by Euc.* 47. 1.) $AB^2 + BC^2 = AC^2$; that is, $b^2 + x^2 = a^2 - 2ax + xx$; whence $x = \frac{a^2 - b^2}{2a} =$ the perpendicular required.

PROBLEM II.

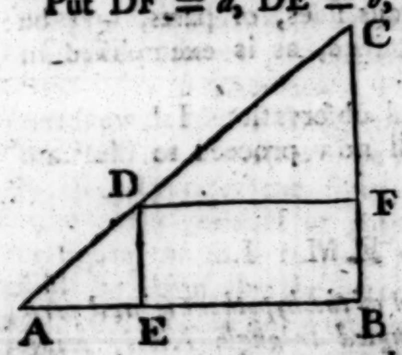
The diagonal, and the perimeter of a rectangle, ABCD, *being given; to find the ſides.*



Put the diagonal $BD = a$, half the perimeter $(DA + AB) = b$, and $AB = x$; then will $AD = b - x$; and therefore, $AB^2 + AD^2$ being $= BD^2$, we have $x^2 + b^2 - 2bx + x^2 = a^2$; which, ſolved, gives $x = \frac{\sqrt{2a^2 - b^2} + b}{2}$.

PROBLEM III.

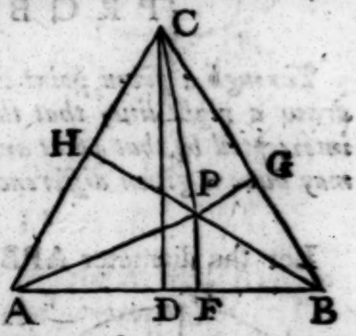
The area of a right-angled triangle ABC, *and the ſides of a rectangle* EBDF *inſcribed therein, being given; to determine the ſides of the triangle.*



Put $DF = a$, $DE = b$, $BC = x$, and the meaſure of the given area $ABC = d$: then, by ſimilar triangles, we ſhall have $x - b$ (CF) : a (DF) :: x (BC) : $AB = \frac{ax}{x - b}$. Therefore $\frac{ax}{x - b} \times \frac{x}{2} = d$, and conſequently $ax^2 = 2dx - 2bd$, or $x^2 - \frac{2dx}{a} = -\frac{2bd}{a}$: which, ſolved, gives $x = \frac{d}{a} \pm \sqrt{\frac{dd}{aa} - \frac{2bd}{a}}$, from whence AB and AC will likewiſe be known.

PROBLEM IV.

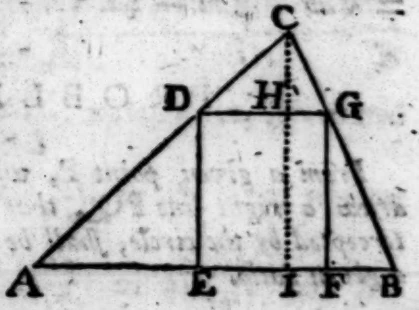
Having the lengths of the three perpendiculars PF, PG, PH, *drawn from a certain point* P *within an equilateral triangle* ABC, *to the three ſides thereof; from thence to determine the ſides.*



Let lines be drawn from P to the three angles of the triangle; and let CD be perpendicular to AB: call PF, a; PG, b; PH, c; and AD $= x$: then will AC ($=$ AB) $= 2x$, and CD ($= \sqrt{AC^2 - AD^2}$) $= \sqrt{3xx} = x\sqrt{3}$; and conſequently the area of the whole triangle ABC ($=$ CD × AD) $= xx\sqrt{3}$. But this triangle is compoſed of the three triangles APB, BPC, and APC; whereof the reſpective areas are ax, bx, and cx. Therefore we have $xx\sqrt{3} = ax + bx + cx$; and from thence, by diviſion, $x = \frac{a + b + c}{\sqrt{3}}$.

PROBLEM V.

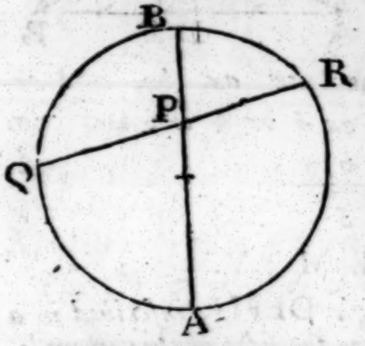
Having the area of a rectangle DEFG, *inſcribed in a given triangle* ABC; *to determine the ſides of the rectangle.*



Let CI be perpendicular to AB, cutting DG in H; and let CI $= a$, AB $= b$, DG $= x$, and the given area $= cc$: then it will be, as $b : x :: a : \frac{ax}{b} =$ CH; which, taken from CI leaves $a - \frac{ax}{b} = IH$; and this, multiplied by x, gives $ax - \frac{ax^2}{b} = cc$ = the area of the rectangle; whence we have $abx - ax^2 = bcc$, $x^2 - bx = -\frac{bcc}{a}$, $x - \frac{b}{2} = \pm\sqrt{\frac{b^2}{4} - \frac{bcc}{a}}$, and $x = \frac{b}{2} \pm \sqrt{\frac{b^2}{4} - \frac{bcc}{a}}$.

PROBLEM VI.

Through a given point P, *within a given circle, ſo to draw a right line, that the two parts thereof,* PR, PQ, *intercepted by that point and the circumference of the circle, may have a given difference.*

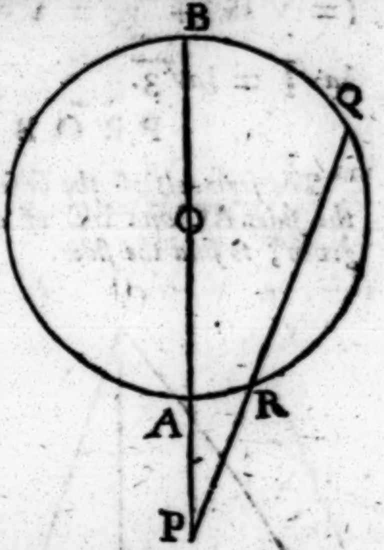


Let the diameter APB be drawn; and let AP and BP, the two parts thereof (which are ſuppoſed given) be denoted by a and b; making PR $= x$, and PQ $= x + d$ (d being the given difference). Then, by the nature of the circle, PQ × PR being = PA × PB, we have $\overline{x + d} \times x = ab$, or $xx + dx = ab$; whence x is found $= \sqrt{ab + \frac{1}{4}dd} - \frac{1}{2}d$.

PROBLEM VII.

From a given point P, *without a given circle, ſo to draw a right line* PQ, *that the part thereof* RQ, *intercepted by the circle, ſhall be to the external part* PR, *in a given ratio.*

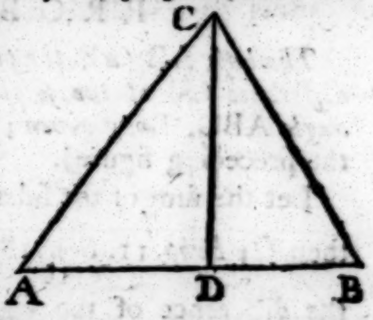
Through the center O, draw PAB; put PA $= a$, PB $= b$, PR $= x$, and let the given ratio of PR to RQ be that of m to n; then it will be, as $m : n :: x : \frac{nx}{m} =$ RQ; therefore PQ $= x + \frac{nx}{m}$: but PR × PQ = PA × PB, or $x \times \overline{x + \frac{nx}{m}} = ab$; therefore $mx^2 + nx^2 = mab$, and $x = \sqrt{\frac{mab}{m+n}}$.



PROBLEM VIII.

The ſum of the two ſides of an iſoſceles triangle ABC *being equal to the ſum of the baſe and perpendicular, and the area of the triangle being given; to determine the ſides.*

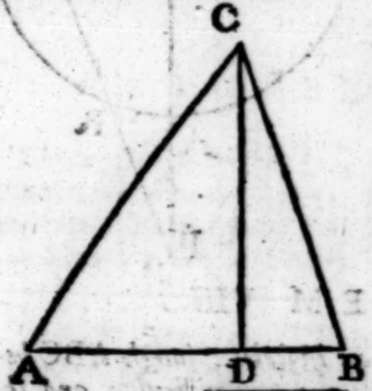
Put the ſemi-baſe AD $= x$, the perpendicular CD $= y$, and the given area ABC $= a^2$: ſo ſhall $xy = a^2$, and $2\sqrt{xx + yy} = 2x + y$ (*by El.* 47. 1. *and the conditions of the problem*). Now, ſquaring both ſides of the laſt equation, we have $4xx + 4yy = 4xx + 4xy + yy$; whence $3yy = 4xy$, and conſequently $y = \frac{4x}{3}$: which value, ſubſtituted in the former equation, gives $\frac{4xx}{3} = a^2$; from whence $x = \sqrt{\frac{3a^2}{4}} = \frac{1}{2}a\sqrt{3}$; $y \left(= \frac{4x}{3}\right) = \frac{2}{3}a\sqrt{3}$; and AC



$(=\sqrt{xx+yy}=\sqrt{\frac{3aa}{4}+\frac{4aa}{3}}=\sqrt{\frac{25aa}{12}})=$ $\frac{5}{2}a\sqrt{\frac{1}{3}}=\frac{5}{6}a\sqrt{3}$.

PROBLEM IX.

The ſegments of the baſe AD *and* BD, *and the ratio of the ſides* AC *and* BC *of any plane triangle* ABC, *being given; to find the ſides.*



Put AD $=a$, BD $=b$, AC $=x$; and let the given ratio of AC to BC, be as m to n, ſo ſhall BC $=\frac{nx}{m}$. But AC^2-AD^2 $(=DC^2)$ $=BC^2-BD^2$, that is, in ſpecies, $x^2-a^2=\frac{nnxx}{mm}-b^2$. Hence we have $m^2x^2-n^2x^2=m^2\times\overline{aa-bb}$, and $x=m\sqrt{\frac{aa-bb}{mm-nn}}$.

PROBLEM X.

The baſe AB *(a), the perpendicular* CD $=b$, *and the difference (d) of the ſides* AC — BC, *of any plane triangle* ABC, *being given; to determine the triangle* (ſee the preceding figure).

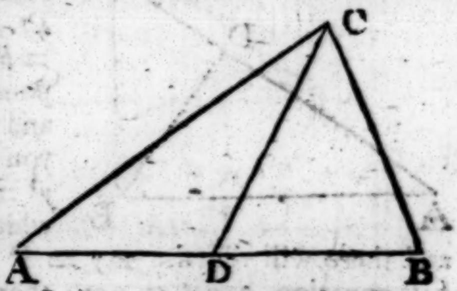
Let the ſum of the ſides AC + BC be denoted by x: then (*by Prop.* 11. *Sect.* 18) we ſhall have $a:x::d:\frac{dx}{a}=$ the difference of the ſegments of the baſe; therefore the greater ſegment AD will be $=\frac{a}{2}+\frac{dx}{2a}=\frac{aa+dx}{2a}$. But $AD^2+DC^2=AC^2$; that is, $\frac{a^4+2a^2dx+d^2x^2}{4aa}+b^2=\frac{x^2+2dx+dd}{4}$: whence

$a^4 + 2a^2dx + d^2x^2 + 4a^2b^2 = a^2x^2 + 2a^2dx + a^2d^2$; which, ſolved, gives $x = a\sqrt{\dfrac{aa + 4bb - dd}{aa - dd}}$.

PROBLEM XI.

The baſe AB, *the ſum of the ſides* AC + BC, *and the length of the line* CD *drawn from the vertex to the middle of the baſe, being given; to determine the triangle.*

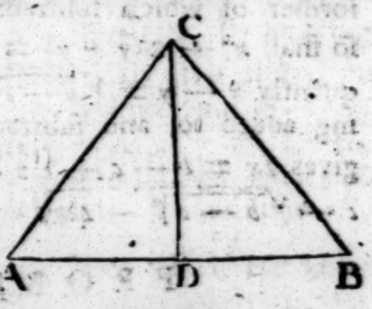
Make AD (= BD) = a, DC = b, AC + BC = c, and AC = x; ſo ſhall BC = $c - x$. But $AC^2 + BC^2$ is = $2AD^2 + 2DC^2$ (*by El.* 12. 2.); that is, $x^2 + \overline{c - x}|^2 = 2a^2 + 2b^2$; which, by reduction, becomes $x^2 - cx = a^2 + b^2 - \frac{1}{2}c^2$; whence x is found $= \frac{1}{2}c \pm \sqrt{aa + bb - \frac{1}{4}cc}$.



PROBLEM XII.

The two ſides AC, BC, *and the line* CD *biſecting the vertical angle of a plane triangle* ABC, *being given; to find the baſe* AB.

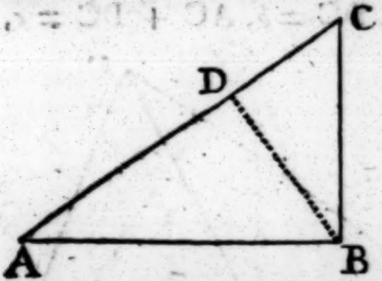
Call AC, a; BC, b; CD, c; and AB, x: then $a + b : x :: a : AD = \dfrac{ax}{a + b}$; and $a + b : x :: b : DB = \dfrac{bx}{a + b}$. But (*by El.* 20. 3.) AC × CB − AD × DB = CD^2, that is $ab - \dfrac{abx^2}{\overline{a + b}|^2} = c^2$; from whence x will be found $= \overline{a + b} \cdot \sqrt{\dfrac{ab - cc}{ab}}$.



PROBLEM XIII.

The perimeter AB + BC + CA, *and the perpendicular* BD, *falling from the right angle* B, *to the hypothenuſe* AC, *being given; to determine the triangle.*

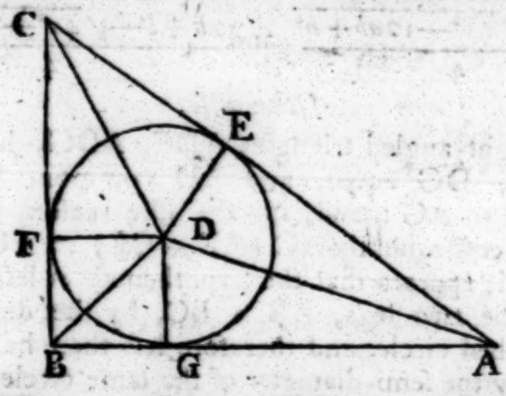
Let BD $= a$, AB $= x$, BC $= y$, AC $= z$, and AB + BC + CA $= b$: then, by reaſon of the ſimilar triangles ACB and ABD, it will be as $z : y :: x : a$; and therefore $xy = az$: moreover, $x^2 + y^2 = z^2$ (*by Euc.* 47. 1.) and $x + y + z = b$ (*by the queſtion*). Tranſpoſe z in the laſt equation, and ſquare both ſides, and you will have $x^2 + 2xy + y^2 = b^2 - 2bz + z^2$; from which take $x^2 + y^2 = z^2$, and there will remain $2xy = b^2 - 2bz$; but, by the firſt equation, $2xy$ is $= 2az$; therefore $2az = b^2 - 2bz$, and $z = \frac{b^2}{2a + 2b}$; whence z is known. But to find x and y from hence, put $\frac{bb}{2a + 2b} = c$, and let this value of z be ſubſtituted in the two foregoing equations, $x + y = b - z$, and $zy = az$, and they will become $x + y = b - c$, and $xy = ac$: from the ſquare of the former of which ſubtract the quadruple of the latter, ſo ſhall $x^2 - 2xy + y^2 = \overline{b - c}|^2 - 4ac$; and conſequently $x - y = \sqrt{\overline{b - c}|^2 - 4ac}$. This equation being added to, and ſubtracted from $x + y = b - c$, gives $2x = b - c + \sqrt{\overline{b - c}|^2 - 4ac}$, and $2y = b - c - \sqrt{\overline{b - c}|^2 - 4ac}$.



PROBLEM XIV.

Having the perimeter of a right-angled triangle ABC, *and the radius* DF, *of its inſcribed circle; to determine all the ſides of the triangle.*

From the centre D, to the angular points, A, B, C, and the points of contact E, F, G, let lines DA, DB, DC, DE, DF, DG be drawn; making DE,



DF, or DG $= a$, AB $= x$, BC $= y$, AC $= z$, and $x + y + z = b$. It is evideat that $\frac{ax}{2} + \frac{ay}{2} + \frac{az}{2}$, or its equal $\frac{ab}{2}$ (expreſſing the ſum of the areas ADB, BDC, and ADC) will be $= \frac{xy}{2} =$ the area of the whole triangle ABC; and conſequently $2xy = 2ab$: moreover (*by Euc.* 47. 1.) $x^2 + y^2 = z^2$; to which if $2xy = 2ab$ be added, we ſhall have $x^2 + 2xy + y^2$, or $\overline{x + y}|^2 = z^2 + 2ab$; but, by the firſt ſtep, $\overline{x + y}|^2$ is $= \overline{b - z}|^2 = b^2 - 2bz + z^2$; therefore, by making theſe two values of $\overline{x + y}|^2$ equal to each other, we get $z^2 + 2ab = b^2 - 2bz + z^2$; whence $2a = b - 2z$, and $z = \frac{1}{2}b - a$. But, to find x and y, from hence, we have now given $x + y \ (= b - z) = \frac{1}{2}b + a$, and $xy = ab$: the former of theſe equations, multiplied by x, gives $x^2 + xy = \frac{bx}{2} + ax$; from which the latter $xy = ab$ being ſubtracted, we have $x^2 = \frac{1}{2}bx + ax - ab$, or $x^2 - \frac{2a + b}{2} \times x = -ab$: whence, by completing the ſquare, &c. $x = \frac{2a + b \pm \sqrt{4a^2 - 12ab + b^2}}{4}$: ſo that the three ſides of the triangle are, $\frac{1}{2}b - a$, $\frac{2a+b+\sqrt{4a^2-12ab+b^2}}{4}$, and $\frac{2a+b-\sqrt{4a^2-12ab+b^2}}{4}$.

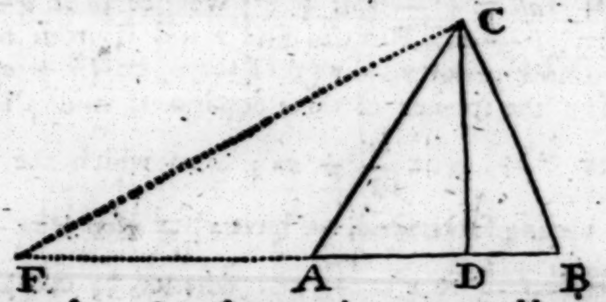
Otherwiſe.

The right-angled triangles ADE, ADG, having the ſides DE, DG equal, and AD common, have alſo AE equal to AG: and, for the like reaſon, is CE = CF; and conſequently AC (AE + CE) = AG + CF. Whence it appears that the hypothenuſe is leſs than the ſum of the two legs, AB + BC, by the diameter of the inſcribed circle, and therefore leſs than half the perimeter by the ſemi-diameter of the ſame circle. Hence we have AC $= \frac{1}{2}b - a$, and AB + BC $= \frac{1}{2}b + a$. Put, therefore, $\frac{1}{2}b - a = c$, $\frac{1}{2}b + a = d$, and half the difference of AB and BC $= x$; then will AB $= d + x$, and BC $= d - x$; and conſequently $2d^2 + 2x^2$ (AB² + BC²) $= c^2$ (AC²), whence x is found $= \sqrt{\frac{1}{2}c^2 - d^2}$; therefore AB is $= \frac{1}{2}d + \sqrt{\frac{1}{2}c^2 - d^2}$, and BC $= \frac{1}{2}d - \sqrt{\frac{1}{2}c^2 - d^2}$.

PROBLEM XV.

All the three ſides of a triangle ABC *being given; to find the perpendicular, the ſegments of the baſe, the area, and the angles.*

Put AC $= a$, AB $= b$, BC $= c$, and the ſegment AD $= x$; then BD being $= b - x$, we have $c^2 - \overline{b - x}|^2$ (= CD²) $= a^2 - x^2$, that is, $c^2 - b^2 + 2bx - x^2 = a^2 - x^2$: whence $2bx = aa + bb - cc$, and



$x =$

$x = \frac{aa+bb-cc}{2b}$: Now $CD^2 = AC^2 - AD^2 =$

$$\overline{AC+AD} \times \overline{AC-AD} = \overline{a + \frac{aa+bb-cc}{2b}} \times \overline{a - \frac{aa+bb-cc}{2b}} = \frac{2ab+aa+bb-cc}{2b} \times \frac{2ab-aa-bb+cc}{2b} = \frac{\overline{a+b}|^2 - c^2}{2b} \times \frac{c^2 - \overline{a-b}|^2}{2b};$$

hence $CD = \frac{1}{2b} \times \sqrt{\overline{\overline{a+b}|^2 - c^2} \times \overline{c^2 - \overline{a-b}|^2}}$;

and the area $\left(\frac{CD \times AB}{2}\right) =$

$$\tfrac{1}{4}\sqrt{\overline{\overline{a+b}|^2 - c^2} \times \overline{c^2 - \overline{a-b}|^2}}.$$

But, becauſe the difference of the ſquares of any two lines, or numbers, is equal to a rectangle under their ſum and difference, the factor $\overline{a+b}|^2 - c^2$ will be $= \overline{a+b+c} \times \overline{a+b-c}$; and the remaining factor $c^2 - \overline{a-b}|^2 = \overline{c+a-b} \times \overline{c-a+b}$: and ſo the area will be likewiſe truly expreſſed by

$$\tfrac{1}{4}\sqrt{\overline{a+b+c} \times \overline{a+b-c} \times \overline{c+a-b} \times \overline{c-a+b}}$$

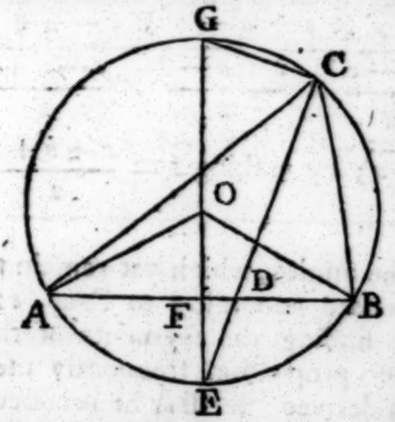
$$= \sqrt{\frac{a+b+c}{2} \times \frac{a+b-c}{2} \times \frac{c+a-b}{2} \times \frac{c-a+b}{2}}$$

$$= \sqrt{s \, . \, \overline{s-c} \, . \, \overline{s-b} \, . \, \overline{s-a}}; \text{ by making } s = \frac{a+b+c}{2}.$$

In order to determine the angles, which yet remain to be conſidered, we may proceed according to Prop. II. in Trigonometry, by firſt finding the ſegments of the baſe: but there is another proportion frequently uſed in practice; which is thus derived: let BA be produced to F, ſo that AF may be = AC; and then, FC being joined, it is plain that the angle F will be the half of the angle A; and DF (= AC + AD) will be given (*from above*) $= \frac{\overline{a+b}|^2 - c^2}{2b} = \frac{a+b+c}{b} \times \frac{a+b-c}{2}$ $= \frac{2s}{b} \times \overline{s-c}$: but DF $(\frac{2s \times \overline{s-c}}{b})$ is to DC $(\frac{2}{b}\sqrt{s.\overline{s-c}.\overline{s-b}.\overline{s-a}})$, ſo is the radius to the tangent of F; and conſequently $s \times \overline{s-c} : \overline{s-b} \times \overline{s-a}$: : ſq. rad. : ſq. tang of F; that is, in words, as the rectangle under half the ſum of the three ſides and the exceſs of that half ſum above the ſide oppoſite the required angle, is to the rectangle under the differences between the other two ſides and the ſaid half ſum, ſo is the ſquare of the radius, to the ſquare of the tangent of half the angle ſought.

PROBLEM XVI.

Having given the baſe AB, *the vertical angle* ACB, *and the right line* CD, *which biſects the vertical angle, and is terminated by the baſe; to find the ſides and angles of the triangle.*



Conceive a circle to be deſcribed about the triangle, and let EG be a diameter of that circle, cutting the baſe AB perpendicularly in F; alſo from the center O, ſuppoſe OA and OB to be drawn, and let CD be produced to E (for it will meet the periphery in that point, becauſe the angles ACD and BCD, being equal, muſt ſtand upon equal arches EA and EB). Now, becauſe the angle AOB at the center, ſtanding upon the arch AEB, is double to the angle ACB at the periphery, ſtanding upon the ſame arch (*Euc.* 20. 3.)

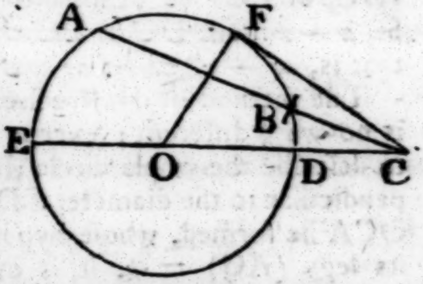
that angle, as well as ACB, is given; and, therefore, in the iſoſceles triangle AOB, there are given all the angles and the baſe AB; whence AO and FO will be both given, by plane trigonometry, and conſequently EF (AO — FO) and EG (= 2AO). Call, therefore, EF = a, EG = b, CD = c, and DE = x; and ſuppoſe CG to be drawn; then, the angle ECG, being a right one (*Euc.* 31. 3.) the triangles EDF and EGC will be ſimilar; whence $x : a :: b : x + c$; therefore, by multiplying extremes and means, we have $x^2 + cx = ab$, and conſequently $x = \sqrt{ab + \frac{1}{4}cc} - \frac{1}{2}c$; from which DF ($\sqrt{ED^2 - EF^2}$), half the difference of the ſegments of the baſe, will be found, and from thence all the reſt, by plane trigonometry.

Before I proceed further in the ſolution of problems, it may not be improper, in order to render ſuch ſolutions more general, to ſay ſomething here, with regard to the geometrical conſtruction of the three forms of adfected quadratic-equations,

$$viz.\begin{cases} x^2 + ax = bc, \\ x^2 - ax = bc, \\ ax - x^2 = bc. \end{cases}$$

Conſtruction of the firſt and ſecond forms.

With a radius equal to $\frac{1}{2}a$, let a circle OAF be deſcribed; in which, from any point A in the periphery, apply AB equal to $b \mathbin{\dot{-}} c$ (b being ſuppoſed greater than c) and produce the ſame 'till BC becomes = c; and from C, thro' the center O, draw CDE cutting the periphery in D and E; then will the value of x be expounded by CD, in the firſt caſe, and by CE, in the ſecond.

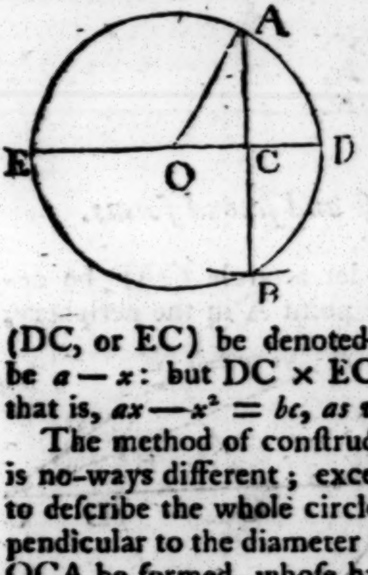


For, ſince (by conſtruction) DE is $= a$, it is plain, if CD be called x, that CE will be $x + a$; but if CE be called x, then CD will be $x - a$: but, *by Eucl.* 37. 3. CE × CD = AC × BC, that is $\overline{x + a} \times x\ (x^2 + ax)$ is $= bc$, in the firſt caſe; and $x \times \overline{x - a}\ (x^2 - ax)$ $= bc$, in the ſecond: which two, are the very equations above exhibited.

When b and c are equal, the conſtruction will be rather more ſimple; for, AB vaniſhing, AC will then coincide with the tangent CF: therefore, if a right-angled triangle OFC be conſtituted, whoſe two legs OF and FC are equal reſpectively to the given quantities $\frac{1}{2}a$ and b, then will CD (= CD — OF) be the true value of x in the former caſe, and CE (= CD + OF) its true value in the latter.

Conſtruction of the third form.

With a radius equal to $\frac{1}{2}a$, let a circle be deſcribed (as in the two preceding forms) in which apply AB, equal to the ſum of the two given quantities $b + c$, and take therein AC equal to either of them; through C draw the diameter DCE; then either DC, or EC, will be the root of the equation.



For, the whole diameter ED being $= a$, it is evident that, if either part thereof (DC, or EC) be denoted by x, the remaining part will be $a - x$: but DC × EC = AC × CB (*Euc.* 35. 3.) that is, $ax - x^2 = bc$, *as was to be ſhewn.*

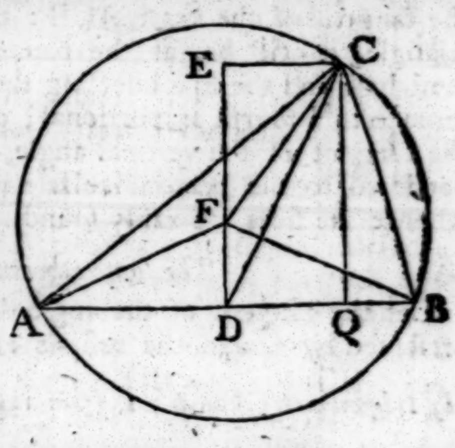
The method of conſtruction, when b and c are equal, is no-ways different; except that it will be unneceſſary to deſcribe the whole circle; for, AC being, here, perpendicular to the diameter ED, if a right-angled triangle OCA be formed, whoſe hypothenuſe is $\frac{1}{2}a$, and one of its legs (AC) $= b$, it is evident that the ſum (EC) and the difference (DC) of the hypothenuſe and the other leg, will be the two values of x required.

Note. If b and c be given ſo unequal, that $b - c$, in the two firſt forms, or $b + c$, in the laſt, exceeds (a) the whole diameter; then, inſtead of thoſe quantities, you may make uſe of any others, as $\frac{1}{2}b$ and $2c$, or $\frac{1}{3}b$ and $3c$, whoſe rectangle, or product is the ſame; or you may find a mean proportional between them, and then proceed according to the latter method.

PROBLEM XVII.

The baſe AB, *the vertical angle* ACB, *and the right line* CD, *drawn from the vertical angle, to biſect the baſe, being given; to find the ſides and perpendicular.*

Suppoſe a circle to be deſcribed about the triangle; and let CQ be perpendicular to AB, and ED equal, and parallel to CQ; moreover, from the center F, let FA, FB, and FC be drawn; alſo let CE be drawn (parallel to AB). Put the ſine of the given angle ACB, to the radius 1, $= m$, its co-ſine $= n$, the ſemi-baſe BD $= a$, the biſecting line CD $= b$, and the perpendicular CQ (DE) $= x$: then, ſince (*by Euc.* 20. 3.) the angle BFD is equal to ACB, it will (by plane trigonometry) be, as m (ſine of BFD) : a (DB) :: n (ſine of DBF) : $\frac{na}{m} =$ DF; and, as m (ſine of BFD) : a (DB) :: 1 (ſine of BDF) : $\frac{a}{m} =$ the radius BF, or FC; whence EF (ED − DF) $= x - \frac{na}{m}$, or $\frac{mx - na}{m}$. But (*by Euc.* 12. 2.) $DF^2 + FC^2 + 2DF \times FE = DC^2$;



that is, in ſpecies, $\frac{n^2a^2}{m^2} + \frac{a^2}{m^2} + \frac{2na}{m} \times \frac{mx - na}{m} = b^2$, or $\frac{a^2}{m^2} - \frac{n^2a^2}{m^2} + \frac{2nax}{m} = b^2$: but, ſince the ſum of the ſquares of the ſine and co-ſine, of any angle whatever, is equal to the ſquare of the radius, or, in the preſent caſe, $m^2 + n^2 = 1$, therefore is $1 - n^2 = m^2$, and conſequently $\frac{a^2}{m^2} - \frac{n^2a^2}{m^2}$, (or $\frac{a^2}{m^2} \times \overline{1 - n^2}$) $= \frac{a^2}{m^2} \times m^2 = a^2$; whence our equation becomes $a^2 + \frac{2nax}{m} = b^2$; which, ordered, gives $x = \frac{m \times \overline{b^2 - a^2}}{2na} = \frac{m}{n} \times \frac{\mathrm{DC}^2 - \mathrm{DB}^2}{\mathrm{AB}} = \frac{m}{n} \times \frac{\overline{\mathrm{DC} + \mathrm{DB}} \times \overline{\mathrm{DC} - \mathrm{DB}}}{\mathrm{AB}}$; where $\frac{m}{n}$ expreſſes the tangent of the angle ACB: therefore, in any plane triangle, it will be, as the baſe is to the ſum of the ſemi-baſe and the line biſecting the baſe, ſo is their difference to a fourth proportional; and, as the radius is to the tangent of the vertical angle, ſo is that fourth proportional to the perpendicular height of the triangle: whence the ſides are eaſily found.

The ſame otherwiſe.

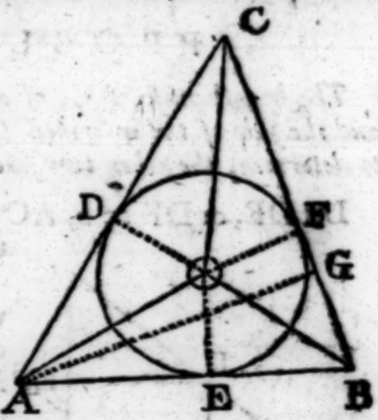
Let the tangent of the angle ACB, or BFD, be repreſented by p, and the reſt as above; then it will be (*by trigonometry*) as $p : 1$ (the radius) $:: a$ (BD) $: \frac{a}{p} = \mathrm{DF}$; therefore FE (DE — DF) $= x - \frac{a}{p}$, and FC^2 $(= \mathrm{FB}^2 = \mathrm{DB}^2 + \mathrm{DF}^2) = a^2 + \frac{a^2}{p^2}$; and conſequently $\frac{a^2}{p^2} + a^2 + \frac{a^2}{p^2} + \frac{2a}{p} \times \overline{x - \frac{a}{p}}$ $(\mathrm{DF}^2 + \mathrm{FC}^2 + 2\mathrm{DF} \times \mathrm{FE}) = bb$ $(= \mathrm{DC}^2)$ that is, $a^2 + \frac{2ax}{p} = b^2$; whence $x = p \times \frac{\overline{b^2 - a^2}}{2a}$, the ſame as before.

PRO-

PROBLEM XVIII.

The area, the perimeter, and one of the angles of any plane triangle ABC, *being given; to determine the triangle.*

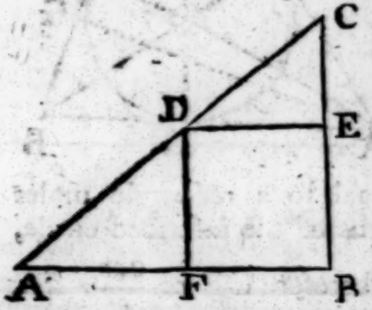
Suppoſe a circle to be inſcribed in the triangle, touching the ſides thereof in the points D, E, and F; alſo from the centre O, ſuppoſe OA, OD, OC, OF, OB, and OE to be drawn: and upon BC let fall the perpendicular AG; putting AB + BC + AC = b, the given area = a^2, the ſine of the angle ACB (the radius being 1) = m, the co-tangent of half that angle (or the tangent of DOC) = n, and AC = x. Therefore, ſince the area of the triangle is equal to $\frac{1}{2}AB \times OE + \frac{1}{2}BC \times OF + \frac{1}{2}AC \times OD$, that is, equal to a rectangle under half the perimeter and the radius of the inſcribed circle, we have $\frac{b}{2} \times OE = aa$; and therefore $OE = \frac{2aa}{b}$. But AD being = AE, and BF = BE; it is manifeſt that the ſum of the ſides, CA + CB, exceeds the baſe AB, by the ſum of the two equal ſegments CD and CF; and ſo is greater than half the perimeter by one of thoſe equal ſegments CD; that is, $CA + CB = \frac{1}{2}b + CD$: but (*by trigonometry*) as 1 (radius) : n (the tangent of DOC) :: $\frac{2aa}{b}$ (OD) : $DC = \frac{2na^2}{b}$; whence $CA + CB\ (= \frac{1}{2}b + CD) = \frac{1}{2}b + \frac{2na^2}{b}$; which, taken from ($b$) the whole perimeter, leaves $\frac{1}{2}b - \frac{2na^2}{b}$ = the baſe AB. Make now $\frac{1}{2}b + \frac{2na^2}{b} = c$; then will $BC = c - x$: alſo (*by trigonometry*) it will be, as 1 (radius) : m (the ſine of



of ACG) : : x (AC) : mx = AG; half whereof, multiplied by $c - x$ (BC), gives $\frac{mcx - mx^2}{2} = a^2$, the area of the triangle: from whence x comes out $= \frac{1}{2}c \pm \sqrt{\frac{c^2}{4} - \frac{2aa}{m}}$.

PROBLEM XIX.

The hypothenuſe, AC, *of a right-angled triangle* ABC, *and the ſide of the inſcribed ſquare* BEDF, *being given; to determine the other two ſides of the triangle.*



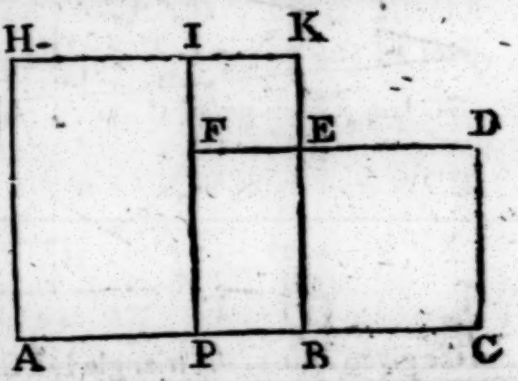
Let DE, or DF $= a$, AC $= b$, AB $= x$, and BC $= y$; then it will be, as $x : y :: x - a$ (AF) : a (FD); whence we have $ax = yx - ya$, and consequently $xy = ax + ay$. Moreover, $xx + yy = bb$: to which equation let the double of the former be added, and there ariſes $x^2 + 2xy + y^2 = b^2 + 2ax + 2ay$; that is, $\overline{x+y}|^2 = b^2 + 2a \times \overline{x+y}$, or $\overline{x+y}|^2 - 2a \times \overline{x+y} = b^2$; where, by conſidering $x + y$ as one quantity, and completing the ſquare, we have $\overline{x+y}|^2 - 2a \times \overline{x+y} + a^2 = b^2 + a^2$; whence $\overline{x+y} - a = \sqrt{b^2 + a^2}$, and $x + y = \sqrt{a^2 + b^2} + a$; which put $= c$: then, by ſubſtituting $c - x$ inſtead of its equal (y) in the equation $xy = ax + ay$, there will ariſe $cx - x^2 = ac$; whence x will be found $= \frac{1}{2}c + \sqrt{\frac{1}{4}cc - ac}$, and $y = \frac{1}{2}c - \sqrt{\frac{1}{4}cc - ac}$.

It appears from hence that c, or its equal $\sqrt{aa + bb} + a$, cannot be leſs than $4a$, and therefore b^2 not leſs than $8a^2$; becauſe the quantity $\frac{1}{4}cc - ac$, under the radical ſign, would be negative, and its ſquare root impoſſible; it being known that all ſquares, whether from poſitive or negative roots, are poſitive; ſo that there cannot ariſe any ſuch things as negative ſquares, unleſs the conditions of the problem under conſideration are inconſiſtent and impoſſible. And this may be demonſtrated, from geometrical principles, by means of the following

LEMMA.

The ſum of the ſquares of any two quantities is greater than a double rectangle under thoſe quantities, by the ſquare of the difference of the ſame quantities.

For let the greater of the two quantities be repreſented by AB, and the leſſer by BC (both taken in the ſame right line). Upon AB and BC let the ſquares AK and CE be conſtituted; take AP = BC and complete the rectangles PH and CF. Therefore, becauſe AB = AH, and AP = BC, it is plain that PH and PD are equal to two rectangles under the propoſed quantities AB and BC; but theſe two rectangles are leſs than the two ſquares AK and CE, which make up the whole figure by the ſquare FK, that is, by the ſquare of PB the difference of the two quantities given; *as was to be proved.*

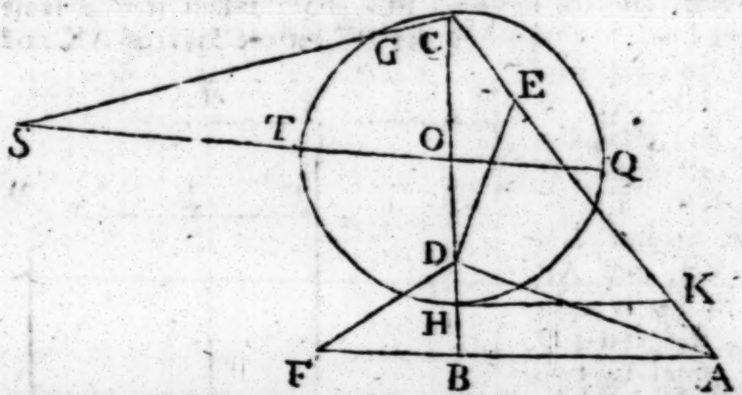


Now, to apply this to the matter propoſed, let there be given the quadratic equation $x^2 + b^2 = 2ax$, or $x = a \pm \sqrt{aa - bb}$: then, I ſay, this equation (and conſequently any problem wherein it ariſes) will be impoſſible, when $aa - bb$ is negative, or b greater than a. For, ſince b is ſuppoſed greater than a, $2bx$ will likewiſe be greater than $2ax$; but $2ax$ is given $= xx + bb$, therefore $2bx$ will be greater than $xx + bb$, that is, the double rectangle of two quantities will be greater than the ſum of their ſquares, *which is proved to be impoſſible.*

PROBLEM XX.

The baſe AB *(a) and the perpendicular* BC *(b) of a right-angled triangle* ABC, *being given; it is propoſed to find a point* D *in the perpendicular, ſo that, if two right-lines be drawn from thence, one to the angular point* A, *and the other* (DE) *perpendicular thereto, the triangles* DEC, ABD, *cut off by thoſe lines, ſhall be to one another in a given ratio.*

Let AB be produced to F ſo that the angle BFD may be equal to the angle BCA; putting AC $= c$, CD $= x$,



and the given ratio of the triangle DEC to ABD, as m to n. Then, by reaſon of the ſimilar triangles ABC, DBF, it will be, a (AB) : b (BC) : : $b - x$ (BD) : BF $= \frac{b^2 - bx}{a}$; whence AF $= a + \frac{b^2 - bx}{a} = \frac{a^2 + b^2 - bx}{a} = \frac{c^2 - bx}{a}$ (becauſe $a^2 + b^2 = c^2$). Alſo, as ADE is a right-angle, the angles FAD, EDC will be equal; therefore, the angles C and F being equal *(by con.)* the triangles AFD, DCE, muſt be ſimilar; and conſequently AF^2 $\left(\frac{\overline{c^2 - bx}|^2}{a^2}\right)$: CD^2 (x^2) : : $\frac{\text{AF} \times \text{BD}}{2}$ $\left(\frac{\overline{b - x} \times \overline{c^2 - bx}}{2a}\right)$ the area of the triangle AFD : $\left(\frac{\overline{b - x} \times ax^2}{2 \times \overline{c^2 - bx}}\right)$ the area of the triangle DEC: wherefore, the area of the triangle

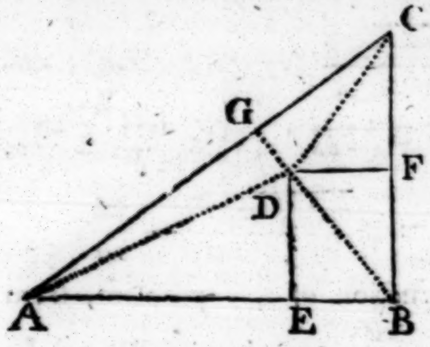
angle ABD being $\frac{BD\times AB}{2}$, or $\frac{\overline{b-x}\times a}{2}$, we ſhall have, $m : n :: \frac{\overline{b-x}\times ax^2}{2\times\overline{c^2-bx}} : \frac{\overline{b-x}\times a}{2}$ (*by the queſtion*); and conſequently, $nx^2 = m\times\overline{c^2-bx}$, or $x^2+\frac{mbx}{n}=\frac{mc^2}{n}$; which, reduced, gives $x=\sqrt{\frac{mc^2}{n}+\frac{m^2b^2}{4n^2}}-\frac{mb}{2n}$.

The geometrical conſtruction of this problem, from the equation $x^2+\frac{mbx}{n}=\frac{mc^2}{n}$, may be as follows. In CB let there be taken, CH : CB : : $m : n$, and let HK be drawn parallel to BA; then CH being $=\frac{mb}{n}$, and CK $=\frac{mc}{n}$, our equation will be changed to $x^2+x\times CH = AC\times CK$, or to $CD\times\overline{CD+CH}=AC\times CK$. Upon CH as a diameter let the circle CTHQ be deſcribed, in which inſcribe CG = AK; and in CG produced, take CS = CA; and from S, through the centre O, draw the right-line STOQ, cutting the circumference in T and Q, and make CD = ST; then will D be the point required. For, CG being = AK, and CS = CA; therefore will $AC\times CK = CS\times GS = ST\times SQ$ (*Euc.* 37. 3.) $= ST\times\overline{ST+TQ} = CD\times\overline{CD+CH}$, the very ſame as above.

PROBLEM XXI.

Having the perimeter of a right-angled triangle ABC, *and three perpendiculars* DE, DF, *and* DG, *falling from a point within the triangle, upon the three ſides thereof; to determine the ſides.*

Suppoſe DA, DB, and DC to be drawn; and let DE $=a$, DF $=b$, DG $=c$, AB $=x$, BC $=y$, AC $=z$, and the perimeter, AB + BC + AC, $= p$: then, the area of ADB being expounded by $\frac{ax}{2}$; that of BDC by $\frac{by}{2}$; that of ADC, by $\frac{cz}{2}$; and that of the whole ABC, by $\frac{xy}{2}$; we therefore have $\frac{ax}{2} + \frac{by}{2} + \frac{cz}{2} = \frac{xy}{2}$, or $ax + by + cz = xy$: moreover, we have $x^2 + y^2 = z^2$, and $x + y + z = p$, by the conditions of the problem. Let z be tranſpoſed in the laſt equation, and both ſides ſquared, ſo ſhall $x^2 + 2xy + y^2 = p^2 - 2pz + z^2$, from which, if $x^2 + y^2 = z^2$ be ſubtracted, there will remain $2xy = p^2 - 2pz = 2ax + 2by + 2cz$ (*by the firſt equation*): whence $ax + by + \overline{c + p} \times z = \frac{1}{2}pp$: from this laſt equation ſubtract a times $x + y + z = p$, and there will remain $by - ay + \overline{p + c - a} \times z = \frac{1}{2}p^2 - ap$; alſo, if from the ſame equation, b times $x + y + z = p$ be ſubtracted, there will remain $ax - bx + \overline{p + c - b} \times z = \frac{1}{2}p^2 - bp$; which two laſt equations, by putting $d = b - a$, $e = p + c - a$, $f = \frac{1}{2}p^2 - ap$, $g = p + c - b$, and $h = \frac{1}{2}p^2 - bp$, will ſtand thus, $dy + ez = f$, and $-dx + gz = h$; whence $y = \frac{f - ez}{d}$, and $x = \frac{gz - h}{d}$. Let theſe values of x and y be ſubſtituted in $x^2 + y^2 = z^2$ and we ſhall have $\frac{f^2 - 2efz + e^2z^2}{d^2} + \frac{g^2z^2 - 2ghz + h^2}{d^2} = z^2$, or $\overline{e^2 + g^2 - d^2} \times z^2 - \overline{2ef + 2gh} \times z = -f^2 - h^2$: put $e^2 + g^2 - d^2 = k$, $ef + gh = l$, and $f^2 + h^2 = m$; ſo ſhall $kz^2 - 2lz = -m$; whence $z^2 - \frac{2lz}{k} = -\frac{m}{k}$, and $z = \frac{l}{k}$



$\pm \sqrt{\frac{l^2}{k^2} - \frac{m}{k}} = \frac{l \pm \sqrt{l^2 - km}}{k}$: from which x ($= \frac{gz - b}{d}$) and y ($= \frac{f - ez}{d}$) will alſo be known.

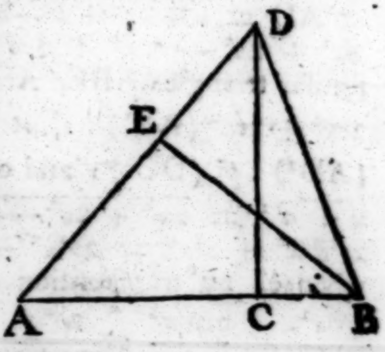
If a, b and c are all equal to each other, the point D will be the centre, and each of the given perpendiculars a radius of the inſcribed circle; and the value of z, in this caſe, will be barely equal to $\frac{1}{2}p - a$; for the equation, $by - ay + \overline{p + c - a} \times z = \frac{1}{2}p^2 - ap$, above found, here becomes $pz = \frac{1}{2}p^2 - ap$.

But, if only a and b (or DE and DF) be equal, then the equation will become $\overline{p + c - a} \times z = \frac{1}{2}p^2 - ap$; and therefore $z = \frac{\frac{1}{2}p^2 - ap}{p + c - a}$; in which, if c be taken $= 0$, z will be $= \frac{\frac{1}{2}p^2 - ap}{p - a}$; where a is the ſide of the inſcribed ſquare.

PROBLEM XXII.

The perpendicular CD, *the difference of the ſides*, AD — BD, *and the vertical angle* D, *of any plane triangle* ABD, *being given; to determine the ſides.*

From B, upon AD (produced if need be) let fall the perpendicular BE: let the ſine of the angle BDE $= s$, its co-ſine $= c$ (the radius being unity); alſo let the perpendicular CD $= p$, the leſſer ſide BD $= x$, and the greater DA $= x + d$: then (*by Prop.* 9. *in trigonometry*) as $1 : s :: x : sx =$ BE; and, as $1 : c :: x : cx =$ ED. Now AB², being $=$ AD² + DB² — AD × 2DE (*Euc.* 13. 2.), will be expounded by $\overline{x + d}|^2 + x^2 - \overline{x + d} \times 2cx$, or $2x^2 + 2dx + d^2 - 2cx^2 - 2cdx$; whence, by reaſon of the ſimilar



ſimilar triangles ABE and ADC, it will be, as $2x^2 + 2dx + d^2 - 2cx^2 - 2cdx$ (AB^2) : s^2x^2 (BE^2) : : $x^2 + 2xd + dd$ (AD^2) : p^2 (DC^2), and conſequently, by multiplying extremes and means, $s^2x^4 + 2s^2dx^3 + s^2d^2x^2 = 2p^2x^2 + 2p^2dx + p^2d^2 - 2p^2cx^2 - 2p^2cdx$; from whence, by tranſpoſition and diviſion, we have $x^4 + \overline{2dx^3 + d^2 - \frac{2p^2}{s^2} + \frac{2p^2c}{s^2}} \times x^2 + \overline{\frac{2p^2cd}{s^2} - \frac{2p^2d}{s^2}} \times x - \frac{p^2d^2}{s^2} = 0$. Which equation anſwering the conditions of the ſecond caſe of biquadratics, explained at *p.* 154, we ſhall therefore have $x^2 + dx + \frac{p^2c - p^2}{s^2} = \pm \sqrt{\frac{p^2d^2}{s^2} + \frac{\overline{p^2c - p^2}|^2}{s^4}}$; and conſequently $x = -\frac{1}{2}d + \sqrt{\frac{d^2}{4} + \frac{p^2 - p^2c}{s^2} + \sqrt{\frac{p^2d^2}{s^2} + \frac{\overline{p^2c - p^2}|^2}{s^4}}}$.

Otherwiſe.

Suppoſing *s*, *c*, and *p* to be the ſame as before, put half the given difference of the ſides $= a$, and half their ſum $= x$; then the greater ſide AD will be $= x + a$, and the leſſer BD $= x - a$; wherefore *(by trigonometry)* $1 : s : : x - a : s \times \overline{x - a} = BE$; and, $1 : c : : x - a : c \times \overline{x - a} = DE$: but AB^2 is $= AD^2 + DB^2 - 2DE \times AD = \overline{x + a}|^2 + \overline{x - a}|^2 - 2c \times \overline{x - a} \times \overline{x + a} = 2x^2 + 2a^2 - 2cx^2 + 2ca^2$; whence, by reaſon of the ſimilar triangles ABE, ADC, it will be $2x^2 + 2a^2 - 2cx^2 + 2ca^2$ (AB^2) : $s^2 \times \overline{x - a}|^2$ (BE^2) : : $\overline{x + a}|^2$ (AD^2) : p^2 (DC^2); and conſequently $s^2 \times \overline{x - a}|^2 \times \overline{x + a}|^2 = \overline{2x^2 + 2a^2 - 2cx^2 + 2ca^2} \times p^2$, or $s^2x^4 - 2s^2a^2x^2 + s^2a^4 = 2p^2x^2 - 2cp^2x^2 + 2p^2a^2 + 2cp^2a^2$; whence, by tranſpoſition and diviſion, $x^4 - 2a^2x^2 - \frac{2p^2x^2}{s^2} + \frac{2cp^2x^2}{s^2} = \frac{2p^2a^2}{s^2} + \frac{2cp^2a^2}{s^2} - a^4$. Subſtitute

$f =$

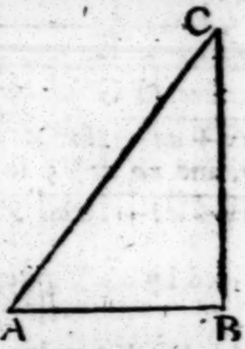
$f = a^2 + \frac{p^2}{s^2} - \frac{cp^2}{s^2}$, and $g = a^2 \times \overline{\frac{2p^2 + 2cp^2}{s^2} - a^2}$; then the equation will ſtand thus, $x^4 - 2fx^2 = g$: whence x is found $= \sqrt{f \pm \sqrt{f^2 + g}}$.

If, inſtead of the difference, the ſum of the ſides had been given, in order to find the difference, the method of operation would have been the very ſame, only, inſtead of finding the value of x in terms of a, by means of the equation $s^2x^4 - 2s^2a^2x^2 + s^2a^4 = 2p^2x^2 - 2cp^2x^2 + 2p^2a^2 + 2cp^2a^2$, that of a muſt have been found in terms of x, from the ſame equation.

PROBLEM XXIII.

Having one leg AB *of a right-angled triangle* ABC; *to find the other leg* BC, *ſo that the rectangle under their difference* (BC — AB) *and the hypothenuſe* AC, *may be equal to the area of the triangle.*

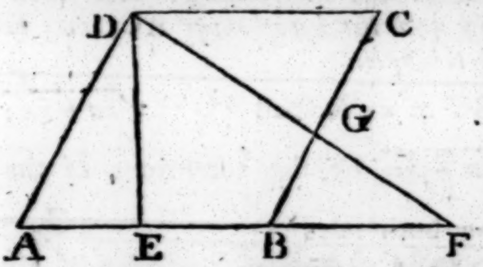
Put AB $= a$, and BC $= x$; ſo ſhall AC $= \sqrt{aa + xx}$; and $\frac{ax}{2} = \overline{x - a}\,\sqrt{aa + xx}$, by the conditions of the problem. By ſquaring both ſides of this equation we have $\frac{1}{4}a^2x^2 = \overline{x^2 - 2qx + a^2} \times \overline{aa + xx}$: in which the quantities x and a being concerned exactly alike, the ſolution will therefore be brought out from the general method for extracting the roots of theſe kinds of equations (delivered at *p.* 156): according to which, having divided the whole by a^2x^2, we get $\frac{1}{4} = \overline{\frac{x}{a} - 2 + \frac{a}{x}} \times \overline{\frac{a}{x} + \frac{x}{a}}$; which, by making $z = \frac{x}{a} + \frac{a}{x}$, will be reduced down to $\frac{1}{4} = \overline{z - 2} \times z$, or $z^2 - 2z = \frac{1}{4}$: whence z is given $= 1 + \sqrt{\frac{5}{4}}$. But, ſince $\frac{x}{a} + \frac{a}{x} = z$, we



we have $x^2 - azx = -a^2$; and therefore $x = \frac{az}{2} + \sqrt{\frac{a^2z^2}{4} - a^2} = \frac{a}{2} \times \overline{z + \sqrt{zz - 4}}$; which, by ſubſtituting the value of z, becomes $x = \frac{a}{2} \times \overline{1 + \sqrt{\frac{5}{4}} + \sqrt{\sqrt{5} - \frac{7}{4}}}$.

PROBLEM XXIV.

To draw a right-line DF *from one angle* D *of a given rhombus* ABCD, *ſo that the part thereof* FG *intercepted by one of the ſides including the oppoſite angle and the other ſide produced, may be of a given length.*

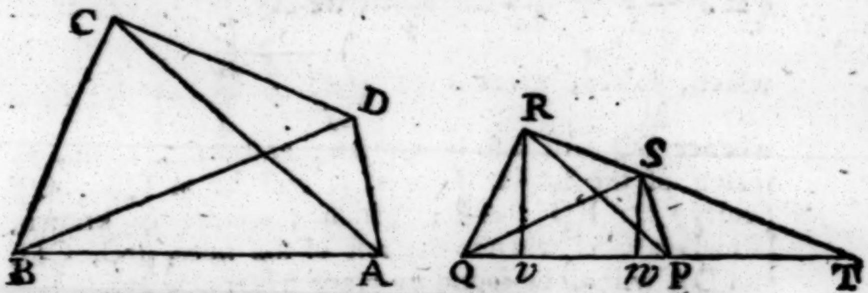


Let DE be perpendicular to AB; and let AB ($=$ AD) $= a$, AE $= b$, FG $= c$, and AF $= x$: then DF^2 ($=$ AF^2 $+$ AD^2 $-$ 2AE$\times$AF) $= xx + aa - 2bx$; and, by ſimilar triangles, $xx + aa - 2bx$(DF^2)$:xx$(AF^2)$::cc$ (FG^2) $: \overline{x - a}^2$ (BF^2); and conſequently $\overline{xx + aa - 2bx} \times \overline{xx - 2ax + aa} = ccxx$. Make $ma = b$, and $na = c$; ſo ſhall our equation become $\overline{xx + aa - 2max} \times \overline{xx - 2ax + aa} = n^2a^2x^2$: which, divided by a^2x^2, gives $\overline{\frac{x}{a} + \frac{a}{x} - 2m} \times \overline{\frac{x}{a} + \frac{a}{x} - 2} = n^2$: this, by making $z = \frac{x}{a} + \frac{a}{x}$, becomes $\overline{z - 2m} \times \overline{z - 2} = n^2$; therefore $z^2 - \overline{2m + 2} \times z = n^2 - 4m$, and $z = 1 + m + \sqrt{n^2 + \overline{1 - m}^2} = \frac{a + b + \sqrt{c^2 + \overline{a - b}^2}}{a}$, by reſtoring the values of m and n. From whence the value of x will be alſo known; for $\frac{x}{a} + \frac{a}{x}$ being $= z$, we have, by reduction, $x^2 - azx = - aa$; and therefore $x = \frac{a}{2} \times \overline{z + \sqrt{zz - 4}}$.

PROBLEM XXV.

The diagonals AC, BD, *and all the angles*, DAB, ABC, BCD, *and* CDA, *of a trapezium* ABCD, *being given; to determine the ſides.*

Let PQRS be another trapezium ſimilar to ABCD, whoſe ſide PQ is unity; and let QP and RS be produced till they meet in T: alſo let PR and QS be drawn, and make Rv and Sw perpendicular to TQ. Let the



(natural) ſine of the given angle STP, to the radius 1, be put $= m$; that of TSP, or PSR, $= n$; that of TRQ $= p$; the co-ſine of SPQ $= r$; that of RQv $= s$; AC $= a$; BD $= b$; and PT $= x$. Then *(by plane trigonometry)* $n : m :: x :$ PS $= \frac{mx}{n}$; and $1 : \frac{mx}{n}$ (PS) $:: r :$ Pw $= \frac{rmx}{n}$: whence, *by Euc.* 13. 2, QS^2 (= QP^2 + PS^2 − 2PQ × Pw) $= 1 + \frac{m^2x^2}{nn} - \frac{2rmx}{n}$.

Again *(by trigonometry)* $p : m :: 1 + x$ (TQ) : QR $= \frac{m + mx}{p}$; and $1 : s :: \frac{m + mx}{p}$ (QR) : Qv $= \frac{ms + msx}{p}$. And therefore PR^2 (= PQ^2 + QR^2 −

$2PQ \times Qv) = 1 + \frac{\overline{m + mx}|^2}{pp} - \frac{2ms + 2msx}{p}$. But, becauſe of the ſimilar figures ABCD, PQRS, it will be, $AC^2 : BD^2 :: PR^2 : QS^2$, that is, $a^2 : b^2 :: 1 + \frac{\overline{m + mx}|^2}{pp} - \frac{2ms + 2msx}{p} : 1 + \frac{m^2x^2}{nn} - \frac{2rmx}{n}$; and conſequently $a^2 + \frac{a^2m^2x^2}{nn} - \frac{2a^2rmx}{n} = b^2 + \frac{b^2m^2}{pp} + \frac{2b^2m^2x}{pp} + \frac{b^2m^2x^2}{pp} - \frac{2b^2ms}{p} - \frac{2b^2msx}{p}$: whence, writing $f = \frac{a^2m^2}{nn} - \frac{b^2m^2}{pp}$, $g = \frac{bbms}{p} - \frac{bbm^2}{pp} - \frac{aarm}{n}$, and $h = b^2 - a^2 + \frac{b^2m^2}{pp} - \frac{2bbms}{p}$, we have $fx^2 + 2gx = h$: which, ſolved, gives $x = \sqrt{\frac{h}{f} + \frac{gg}{ff}} - \frac{g}{f}$: from whence SQ will alſo be known: and then, again, by reaſon of the ſimilar figures, it will be as QS : QP (unity) : : BD : AB; which therefore is known, likewiſe: from whence the reſt of the ſides, BC, CD, and DA will all be found by plane trigonometry.

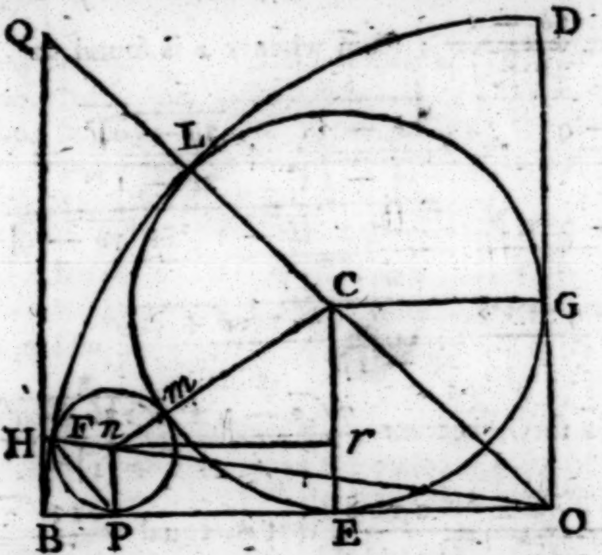
The laſt problem is indeterminate in that particular caſe, where the trapezium may be inſcribed in a circle, or where the ſum of the two oppoſite angles is equal to two right ones; for, then, there can but one diagonal be given, in the queſtion, becauſe the value of the other depends intirely upon that.

PROBLEM XXVI.

Suppoſing BOD *to be a quadrant of a given circle; to find the ſemi-diameter* CE, *or* CL, *of the circle* CEGL *inſcribed therein; and likewiſe the ſemi-diameter of the little circle* nFmP, *touching both the other circles* DLB, LmE, *and alſo the right-line* OB.

Let BQ, P*n*, and CE, be perpendicular to BO; join C, *n* and O, *n*; and draw OC meeting BQ in Q, and *nr* parallel to BO, meeting CE in *r*: put OB (=

(= BQ) = 1, OQ (= $\sqrt{2}$ *by Euc.* 47. 1.) = b, and Pn (= nm) = x. Then, by reaſon of the ſimilar triangles OBQ, OEC, it will be, OQ : BQ : : OC :



CE; whence, *by compoſition*, OQ + BQ : BQ : : OL (OC + CE) : CE; that is, $b + 1 : 1 : : 1 : \text{CE} =$

$$\frac{1}{b+1}\left(=\frac{b-1}{\overline{b+1}\times\overline{b-1}}\right)=\frac{b-1}{b^2-1}=b-1=\sqrt{2}$$

-1; which let be denoted by a, then we ſhall have Cn + Cr = $2a$, and Cn — Cr = $2x$; and therefore nr $(=\sqrt{\overline{Cn+Cr}\times\overline{Cn-Cr}})=2\sqrt{ax}$. Moreover O$n$ + Pn being = 1, and On — Pn = $1-2x$, thence will OP = $\sqrt{1-2x}$; which alſo being = PE + OE $(2\sqrt{ax}+a)$ we therefore have $\sqrt{1-2x}=2\sqrt{ax}+a$; whereof both ſides being ſquared, there ariſes $1-2x=4ax+4a\sqrt{ax}+a^2$, or $1-a^2-2x-4ax=4a\sqrt{ax}$; which, becauſe $1-aa$ is $2a$, will be $a-\overline{1+2a}\times x=2a\sqrt{ax}$: this, ſquared, gives $a^2-\overline{1+2a}\times 2ax+\overline{1+2a}^2\times x^2=4a^3x$; whence $\overline{1+2a}^2\times x^2-\overline{1+2a}\times 2ax-4a^3x=-aa$; which, by writing

$b-$

$b-1$ inſtead of its equal a, becomes $\overline{2b-1}|^2 \times x^2 - \overline{7b-9} \times 2x = 2b-3$; therefore $x^2 - \frac{7b-9}{\overline{2b-1}|^2} \times 2x = \frac{2b-3}{\overline{2b-1}|^2}$; from whence x is found $=$

$$\frac{7b-9}{\overline{2b-1}|^2} \pm \sqrt{\frac{2b-3}{\overline{2b-1}|^2} + \frac{\overline{7b-9}|^2}{\overline{2b-1}|^4}} =$$

$$\frac{7b-9 \pm \sqrt{\overline{2b-3} \times \overline{2b-1}|^2 + \overline{7b-9}|^2}}{\overline{2b-1}|^2} =$$

$\frac{7b-9 \pm \sqrt{8b^3+29b^2-112b+78}}{\overline{2b-1}|^2}$: which, by writing in $\sqrt{2}$ for b, becomes $\frac{7\sqrt{2}-9 \pm \sqrt{136-96\sqrt{2}}}{\overline{2\sqrt{2}-1}|^2} =$ $\frac{7\sqrt{2}-9 \pm 6\sqrt{2} \mp 8}{\overline{2\sqrt{2}-1}|^2}$, that is, equal to $\frac{13\sqrt{2}-17}{\overline{2\sqrt{2}-1}|^2}$, or to $\frac{\sqrt{2}-1}{\overline{2\sqrt{2}-1}|^2}$; which laſt is the root required, the other being manifeſtly too large: but this value will be reduced to $\frac{5\sqrt{2}-1}{49}$. Therefore OP $(=\sqrt{1-2x})$ is given $=\sqrt{\frac{51-10\sqrt{2}}{49}} = \frac{5\sqrt{2}-1}{7} = 7Pn$; and conſequently BH $= \frac{1}{7}$BQ; from whence we have the following conſtruction.

In the tangent BQ, take BH $= \frac{1}{7}$BO; draw HO, cutting the circumference BDL in F, and make the angle OFP $= \frac{1}{2}$ OHB, and draw Pn parallel to BQ, meeting OH in n, the center of the leſſer circle required.

SCHOLIUM.

In the preceding ſolution it was required, not only to extract the ſquare root of the radical quantities 136 —

$96\sqrt{2}$ and $51 - 10\sqrt{2}$, but likewiſe to take away the radical quantity from the denominator of the fraction $\frac{\sqrt{2}-1}{\overline{2\sqrt{2}-1}|^{2}}$, and confine it, wholly, to the numerator: all which being ſomewhat difficult, (and, for that reaſon, omitted in the introduction, as too diſcouraging to a young beginner) I ſhall therefore take the opportunity to explain *here* the manner of proceeding in ſuch like caſes, when they happen to occur.

Firſt, then, with regard to the extraction of roots out of radical quantities, let there be propoſed $A \pm \sqrt{B}$, A being the rational, and $\sqrt{B}$ the irrational part thereof; and let the root required be repreſented by $\sqrt{x} \pm \sqrt{y}$; the ſquare of which will be $x + y \pm 2\sqrt{xy}$, or $x + y \pm \sqrt{4xy}$; therefore we have $x + y \pm \sqrt{4xy} = A \pm \sqrt{B}$. Let the irrational, as well as the rational parts of theſe two equal quantities be now compared together; ſo ſhall $x + y = A$, and $\sqrt{4xy} = \sqrt{B}$: from the ſquare of the former of which equations ſubtract that of the latter, whence will be had $xx - 2xy + yy = A^{2} - B$; and, by taking the ſquare root, $x - y = \sqrt{A^{2} - B}$; which added to, and ſubtracted from $x + y = A$, &c. gives $x = \frac{A + \sqrt{A^{2} - B}}{2}$, and $y = \frac{A - \sqrt{A^{2} - B}}{2}$.

In the firſt of the two caſes above ſpecified, the quantity whoſe ſquare root is to be extracted being $136 - 96\sqrt{2}$, we have $A = 136$, and $B = 18432$ ($= \overline{96}|^{2} \times 2$); whence we have x ($= \frac{A + \sqrt{A^{2} - B}}{2}$) $= 72$, and y ($= \frac{A - \sqrt{A^{2} - B}}{2}$) $= 64$; and conſequently $\sqrt{x} - \sqrt{y} = \sqrt{72} - \sqrt{64} = 6\sqrt{2} - 8$, the required ſquare root of $136 - 96\sqrt{2}$. After the very ſame manner, the ſquare root of the other radical quantity $51 - 10\sqrt{2}$, or $51 - \sqrt{200}$ will be found to be $5\sqrt{2} - 1$:

for, A being here $= 51$, and B $= 200$, we have $x = 50$; and $y = 1$; and conſequently $\sqrt{x} - \sqrt{y} = 5\sqrt{2} - 1$.

What has been ſaid, thus far, relates to the extraction of the ſquare root only; but the ſame method is eaſily extended to the cube, biquadratic, or any other root. Let us take an inſtance thereof in the cube root; where we will ſuppoſe the given quantity, out of which the root is to be extracted, to be repreſented by $A \pm \sqrt{B}$, as before. Then, if the rational part of the root be denoted by x, and the irrational part by $\sqrt{y}$; the root itſelf will be expreſſed by $x \pm \sqrt{y}$; and its cube by $x^3 \pm 3x^2\sqrt{y} + 3xy \pm y\sqrt{y}$: from whence, by proceeding as in the extraction of the ſquare root, we have $x^3 + 3xy = A$, and $3x^2\sqrt{y} + y\sqrt{y} = \sqrt{B}$. Let the ſum and the difference of theſe two equations be taken, and there will come out $x^3 + 3x^2\sqrt{y} + 3xy + y\sqrt{y} = A + \sqrt{B}$, and $x^3 - 3x^2\sqrt{y} + 3xy - y\sqrt{y} = A - \sqrt{B}$; whereof the cube root being extracted on both ſides, we thence have $x + \sqrt{y} = \overline{A + \sqrt{B}}\rceil^{\frac{1}{3}}$, and $x - \sqrt{y} = \overline{A - \sqrt{B}}\rceil^{\frac{1}{3}}$: let the two laſt equations be added together, and the ſum be divided by 2; ſo ſhall $x = \dfrac{\overline{A + \sqrt{B}}\rceil^{\frac{1}{3}} + \overline{A - \sqrt{B}}\rceil^{\frac{1}{3}}}{2}$: and by multiplying the ſame equations together, we get $x^2 - y = \overline{A^2 - B}\rceil^{\frac{1}{3}}$, and conſequently $y = x^2 - \overline{A^2 - B}\rceil^{\frac{1}{3}}$; whence y is likewiſe known.

Univerſally, let the index of the root to be extracted be denoted by n, and let the root itſelf be repreſented by $x \pm \sqrt{y}$ (as above). Then this expreſſion, raiſed to the nth power, will be $x^n \pm nx^{n-1}\sqrt{y} + n \times \dfrac{n-1}{2}x^{n-2}y \pm n \times \dfrac{n-1}{2} \times \dfrac{n-2}{3}x^{n-3}y\sqrt{y}$ &c. from whence, ſtill following the ſame method, we ſhall here get $x^n + n \times \dfrac{n-1}{2}x^{n-2}y$ &c. $= A$, and

$nx^{n-1}\sqrt{y}+n\times\frac{n-1}{2}\times\frac{n-2}{3}x^{n-2}y\sqrt{y}$ &c. $=\sqrt{B}$: let, therefore, the root of the ſum, and alſo of the difference of theſe two laſt equations, be taken, and you will have, $x+\sqrt{y}=\overline{A+\sqrt{B}}|^{\frac{1}{n}}$, and $x\backsim\sqrt{y}=\overline{A-\sqrt{B}}|^{\frac{1}{n}}$; which two equations being added together, x will be found $=\frac{\overline{A+\sqrt{B}}|^{\frac{1}{n}}\pm\overline{A-\sqrt{B}}|^{\frac{1}{n}}}{2}$ $=\frac{\overline{A+\sqrt{B}}|^{\frac{1}{n}}}{2}\pm\frac{\overline{A^2-B}|^{\frac{1}{n}}}{2\times\overline{A+\sqrt{B}}|^{\frac{1}{n}}}$: and, if the ſame equations be multiplied together, you will have $x^2\backsim y=\overline{A^2-B}|^{\frac{1}{n}}$; whence $y=x^2\pm\overline{A^2-B}|^{\frac{1}{n}}$: The uſe of which concluſions will appear by the following examples.

Firſt, let it be propoſed to extract the cube root of the radical quantity $26+15\sqrt{3}$, or $26+\sqrt{675}$. Here, A being $=26$, $B=675$, and $n=3$, we have $x\left(=\frac{\overline{26+\sqrt{675}}|^{\frac{1}{3}}}{2}\pm\frac{1}{2\times\overline{26+\sqrt{675}}|^{\frac{1}{3}}}=\frac{3.732\pm.268}{2}\right)$ $=2$; and $y\left(=4-\overline{676-675}|^{\frac{1}{3}}\right)=3$; and conſequently $x+\sqrt{y}=2+\sqrt{3}=$ the value required; for $\overline{2+\sqrt{3}}\times\overline{2+\sqrt{3}}\times\overline{2+\sqrt{3}}=26+\sqrt{675}$.

Again, let it be required to extract the biquadratic root of $161+\sqrt{25920}$. In this caſe, A being $=161$, $B=25920$, and $n=4$, we have $x\left(=\frac{\overline{321.996896}|^{\frac{1}{4}}}{2}\pm\frac{1}{2\times\overline{321,99\,\&c.}|^{\frac{1}{4}}}=\frac{4.236\pm.236}{2}\right)=2$, and y (=

$y\ (= 4 + \overline{25921 - 25920}|^{\frac{1}{2}}) = 5$; therefore the root ſought is, here, $= 2 + \sqrt{5}$.

Laſtly, if it were required to find the firſt ſurſolid root of $76 + \sqrt{5808}$; then, by proceeding in the ſame manner, x will be found $(= \frac{2,732 \mp ,732}{2}) = 1$, and $y\ (= 1 - \overline{5776 - 5808}|^{\frac{1}{5}}) = 3$: and ſo of others. But it is to be obſerved, that the ſecond part of the value of x, to which both the ſigns + and — are prefixed, is to be taken affirmative or negative, according as *that* or *this* ſhall be found requiſite to make the value of x come out a whole, or rational, number; and that, if neither of the ſigns give ſuch a value of x, then this method is of no uſe, and we may ſafely conclude that the quantity propoſed does not admit of ſuch a root as we would find. It may alſo be proper to remark here, that, if the upper ſign in the value of x be taken, the upper ſign in that of y muſt be taken accordingly; and that the application of logarithms will be of uſe to facilitate the extraction of the root $\overline{A + \sqrt{B}}|^{\frac{1}{n}}$, as being ſufficiently exact to determine whether x be a whole number, and, if ſo, what it is.

Thus much in relation to the extraction of the roots of radical quantities; it remains now to explain the manner of taking away radical quantities out of the denominator of a fraction, and tranſplanting them into the numerator.

In order to which, ſuppoſing r to denote a whole number, it is evident, in the firſt place, that $x^r - y^r = \overline{x - y} \cdot \times \overline{x^{r-1} + x^{r-2}y + x^{r-3}y^2 \ldots + y^{r-1}}$; ſince, by an actual multiplication, the product appears to be $\left\{\begin{matrix} x^r + x^{r-1}y + x^{r-2}y^2 + x^{r-3}y^3 \&c. \\ - x^{r-1}y - x^{r-2}y^2 - x^{r-3}y^3 \ldots - y^r \end{matrix}\right\}$, where all the terms, except the firſt and laſt, deſtroy one another. Hence, by equal diviſion, we have

$$\frac{1}{x-y}=\frac{x^{r-1}+x^{r-2}y+x^{r-3}y^{2}\ldots+y^{r-1}}{x^{r}-y^{r}}.$$ And, in the very ſame manner, it will appear that

$$\frac{1}{x+y}=\frac{x^{r-1}-x^{r-2}y+x^{r-3}y^{2}-x^{r-4}y^{3}\ldots\pm y^{r-1}}{x^{r}\pm y^{r}}:$$

where the ſign — or +, in the denominator, takes place, according as the number r is even or odd.

Let now $x=A^{m}$, and $y=B^{n}$; then our equations will become

$$\frac{1}{A^{m}-B^{n}}=\frac{A^{rm-m}+A^{rm-2m}B^{n}+A^{rm-3m}B^{2n}\ldots\ldots+B^{rn-n}}{A^{rm}-B^{rn}};$$

$$\frac{1}{A^{m}+B^{n}}=\frac{A^{rm-m}-A^{rm-2m}B^{n}+A^{rm-3m}B^{2n}\ldots\ldots\pm B^{rn-n}}{A^{rm}\mp B^{rn}}.$$

From which theorems, or general *formulæ*, the matter propoſed to be done, may be effected with great facility: for, ſuppoſing $\frac{1}{A^{m}-B^{n}}$, or $\frac{1}{A^{m}+B^{n}}$ to be a fraction having radical quantities A^{m}, B^{n} in the denominator, it is plain, that its equal value, given by the ſaid equations, will have its denominator intirely free from radical quantities, if r be ſo aſſumed that both rm and rn may be integers.

To exemplify which, let the fraction $\frac{1}{\sqrt{2}-1}$, or $\frac{1}{\overline{2}|^{\frac{1}{2}}-1}$ be propounded; then, A being $=2$, $B=1$, $m=\frac{1}{2}$ and $n=1$, we ſhall, by taking $r=2$, have (*from Theorem* 1) $\frac{1}{\overline{2}|^{\frac{1}{2}}-1}=\frac{\overline{2}|^{\frac{1}{2}}+1}{2-1}=\sqrt{2}+1$.

Again, let the given fraction be $\frac{1}{\sqrt{cx} + \sqrt[4]{c^4 + x^4}}$, or $\frac{1}{\overline{cx}|^{\frac{1}{2}} + \overline{c^4 + x^4}|^{\frac{1}{4}}}$. In which caſe, A being $= cx$, B $= c^4 + x^4$, $m = \frac{1}{2}$, and $n = \frac{1}{4}$, we ſhall, by taking $r = 4$, have $\frac{1}{\overline{cx}|^{\frac{1}{2}} + \overline{c^4 + x^4}|^{\frac{1}{4}}} =$

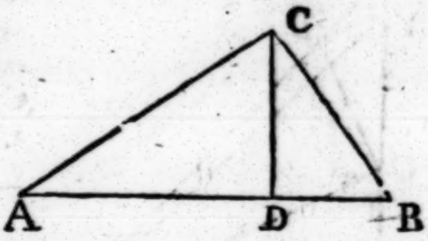
$$\frac{\overline{cx}|^{\frac{3}{2}} - cx \times \overline{c^4 + x^4}|^{\frac{1}{4}} + cx^{\frac{1}{2}} \times \overline{c^4 + x^4}|^{\frac{1}{2}} - \overline{c^4 + x^4}|^{\frac{3}{4}}}{c^2 x^2 - c^4 - x^4}.$$

If the numerator is not an unit, you may proceed in the ſame manner, and multiply *afterwards* by the numerator given. Thus, in the caſe mentioned at the beginning of this ſcholium, we had given $\frac{\sqrt{2} - 1}{\overline{2\sqrt{2} - 1}|^2}$, which may be reduced to $\overline{\sqrt{2} - 1} \times \frac{1}{2\sqrt{2} - 1} \times \frac{1}{2\sqrt{2} - 1}$, or to $\overline{\sqrt{2} - 1} \times \frac{1}{8^{\frac{1}{2}} - 1} \times \frac{1}{8^{\frac{1}{2}} - 1}$: but $\frac{1}{8^{\frac{1}{2}} - 1}$ is found (*by Theorem* 1) o be $= \frac{\sqrt{8} + 1}{8 - 1} = \frac{2\sqrt{2} + 1}{7}$: whence our expreſſion becomes $\overline{\sqrt{2} - 1} \times \frac{2\sqrt{2} + 1}{7} \times \frac{2\sqrt{2} + 1}{7}$: which, by multiplication, &c. is reduced, at length, to $\frac{5\sqrt{2} - 1}{49}$.

PROBLEM XXVII.

Having one leg AC, *of a right-angled triangle* ABC, *to find the other leg* BC, *ſo that the hypothenuſe* AB *ſhall be a mean proportional between the perpendicular* CD *falling thereon, and the perimeter of the triangle.*

Put AC $= a$, and BC $= x$; then will AB $= \sqrt{xx + aa}$, and CD $\left(= \frac{AC \times BC}{AB}\right) = \frac{ax}{\sqrt{xx + aa}}$: therefore, *by the queſtion*, $x + a + \sqrt{xx + aa} : \sqrt{xx + aa} :: \sqrt{xx + aa} : \frac{ax}{\sqrt{xx + aa}}$; and conſequently $\frac{ax^2 + a^2x}{\sqrt{xx + aa}} + ax = xx + aa$: whence $a^2x^4 + 2a^3x^3 + a^4x^2 = \overline{xx + aa - ax}|^2 \times \overline{xx + aa}$. Divide by a^3x^3 (according to the rule at *page* 156) ſo ſhall $\frac{x}{a} + 2 + \frac{a}{x} = \overline{\frac{x}{a} + \frac{a}{x} - 1}|^2 \times \overline{\frac{x}{a} + \frac{a}{x}}$: which, by making $z = \frac{x}{a} + \frac{a}{x}$, is reduced to $z + 2 = \overline{z - 1}|^2 \times z$, or $z^3 - 2z^2 = 2$. This, ſolved (by the rule for cubics) gives $z = \frac{2}{3} + \frac{1}{3} \times \overline{35 + \sqrt{1161}}|^{\frac{1}{3}} + \frac{1}{3} \times \frac{4}{\overline{35 + \sqrt{1161}}|^{\frac{1}{3}}} = 2{,}359304$: whence $x \left(= \frac{a}{2} \times z \pm \sqrt{zz - 4}\right)$ will likewiſe be known.

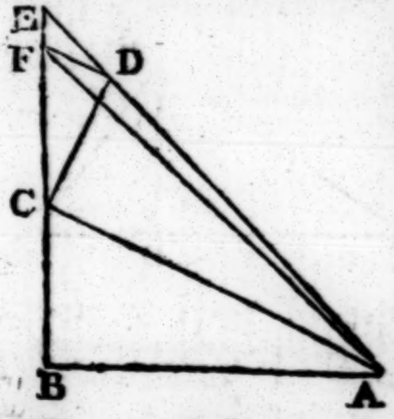


PROBLEM XXVIII.

The baſe AB, *and the pependicular* BE, *of a right-angled triangle* ABE *being given; it is propoſed to find a point* (C) *in the perpendicular, from whence two right-lines* CA *and* CD *being drawn, at right-angles to each other, the two triangles* ACD *and* ABC, *formed from thence, ſhall be equal.*

Suppoſe DF parallel to AC, and let AF be drawn: putting AB $= a$, BE $= b$, BC $= x$, and AC $(\sqrt{a^2 + x^2}) = y$.

$= y$. Then, ſince FD is parallel AC, the triangle, ACF will be equal to ACD, or ABC; and therefore CF $=$ BC $= x$; whence we have EF ($=$ EB $-$ BF) $= b - 2x$, and EC ($=$ EB $-$ BC) $= b - x$: moreover by reaſon of the ſimilar triangles ABC and CDF, we have, y (AC) : x (BC) : : x (CF) : FD $= \frac{xx}{y}$.



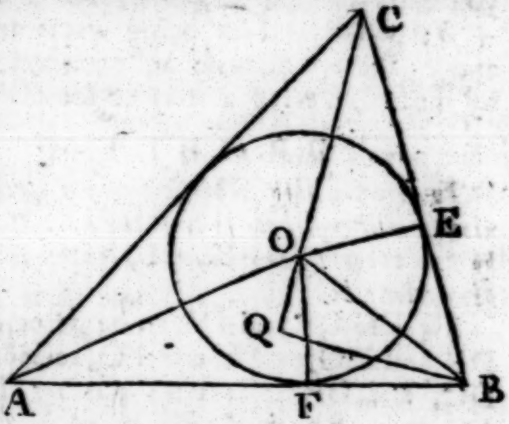
Whence, becauſe of the parallel lines AC and FD, it will be, $\frac{x^2}{y}$ (FD) : $b - 2x$ (EF) : : y (AC) : $b - x$ (CE); and conſequently $\frac{bx^2 - x^3}{y} = \overline{b - 2x} \times y$, or $bx^2 - x^3 = \overline{b - 2x} \times y^2$; which equation, by writing $a^2 + x^2$, inſtead of its equal, y^2, becomes $bx^2 - x^3 = ba^2 + bx^2 - 2a^2x - 2x^3$; whence we have $x^3 + 2a^2x = ba^2$, and therefore $x =$

$$\sqrt[3]{\frac{ba^2}{2} + a^2\sqrt{\frac{b^2}{4} + \frac{8a^2}{27}}} - \frac{\frac{2}{3}aa}{\sqrt[3]{\frac{baa}{2} + a^2\sqrt{\frac{bb}{4} + \frac{8aa}{27}}}}.$$

PROBLEM XXIX.

Three lines AO, BO, CO, *drawn from the angular points of a triangle to the center of the inſcribed circle, being given; to find the radius of the circle and the ſides of the triangle.*

If, upon CQ produced, the perpendicular BQ be let fall, and the radii OE, OF be drawn to the points of contact, the triangles BOQ and AOF will appear to be equiangular; becauſe all the angles of the triangle ABC being equal to two right ones, the ſum of all their halves, OCB + OBC + OAF will be equal to one right-angle; but the two former of theſe, OCB + OBC, is equal to the external angle QOB; therefore QOB + OAF = a right-angle = QOB + OBQ, and conſequently OAF = OBQ. Put now AO = a, BO = b, CO = c, and OF = x: then, becauſe of the ſimilar triangles, we have $a : x :: b : \mathrm{OQ} = \frac{bx}{a}$; whence BQ^2 $= b^2 - \frac{bbxx}{aa}$, and BC^2 (= CO^2 + BO^2 + 2CO × OQ) $= b^2 + c^2 + \frac{2bcx}{a}$. But BC^2 : BQ^2 :: OC^2 : OE^2; that is, $\frac{ab^2 + ac^2 + 2bcx}{a} : \frac{a^2b^2 - b^2x^2}{aa} :: c^2 : x^2$. From whence we get this equation, *viz.* $ax^2 \times \overline{ab^2 + ac^2 + 2bcx} = b^2c^2 \times \overline{a^2 - x^2}$; which, by reduction, will become $x^3 + \overline{\frac{ab}{2c} + \frac{ac}{2b} + \frac{bc}{2a}} \times x^2 = \frac{abc}{2}$: whence x may be found, and from thence the ſides of the triangle.



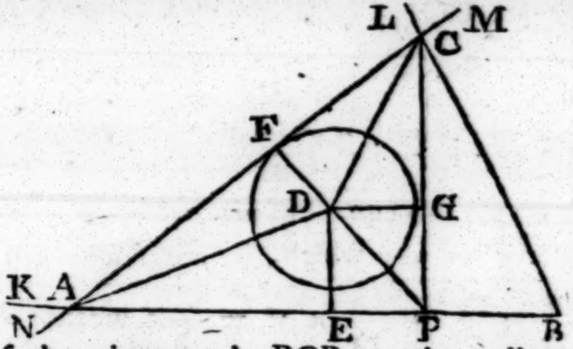
If two of the given lines, as OC and OB, be ſuppoſed equal, the reſult will be more ſimple: for, by writing b for c in the equation $ax^2 \times \overline{ab^2 + ac^2 + 2bcx} = b^2c^2 \times \overline{a - x}$, &c. we ſhall have $ab^2x^2 \times \overline{2a + 2x} = b^4 \times \overline{a - x^2}$; which, divided by $b^2 \times \overline{a + x}$, gives $2ax^2 = b^2 \times \overline{a - x}$; whence $x^2 + \frac{b^2x}{2a} = \frac{1}{2}b^2$, and $x = \sqrt{\frac{b^2}{2} + \frac{b^4}{16aa}} - \frac{b^2}{4a} = \frac{b\sqrt{8aa + bb} - bb}{4a}$.

From the ſame equations the problem may be reſolved, when the diſtances from the three angular points to the circumference of the inſcribed circle are given: for, denoting the ſaid diſtances by f, g, and h, you will have $AO = x + f$, $BO = x + g$, and $CO = x + h$; which values being wrote in the room of a, b, and c, there will ariſe an equation of ſix dimenſions: by means whereof x may be found.

PROBLEM XXX.

To draw a line NM *to touch a circle* D, *given in magnitude and poſition, ſo that the part thereof* AC, *intercepted by two other lines* BK, BL, *given in poſition, ſhall be of a given length.*

Suppoſe CP and DE to be perpendicular to AB, and DF and DG to AC and PC, reſpectively; and let DA, DC, and DP be drawn; putting $DE = a$, $DF = b$, $AC = c$, $BE = d$, $PC = x$, $PA = y$, and the tangent of the given angle BCP, to the radius $1, = t$.



Then, *by trigonometry*, $1 : t :: x : tx = BP$; therefore $DG\ (= PE) = d - tx$; which, multiplied by $\frac{1}{2}PC$, or $\frac{x}{2}$, gives $\frac{dx - tx^2}{2}$, for the area of the triangle CDP: in like manner the area of the triangle PDA will be found $= \frac{ay}{2}$; and *that* of $ADC = \frac{bc}{2}$; which three, added together, are equal to the whole area ACP; that is, $\frac{dx - tx^2}{2} + \frac{ay}{2} + \frac{bc}{2} = \frac{xy}{2}$; and conſequently $bc + dx - tx^2 = xy - ay$. Let both ſides of this equation be ſquared, and you will have $\overline{bc + dx - tx^2}\rceil^2 = \overline{x - a}\rceil^2 \times y^2 = \overline{x - a}\rceil^2 \times \overline{c^2 - x^2}$; that is, $b^2c^2 + 2bcdx - 2bctx^2 + d^2x^2 - 2dtx^3 + t^2x^4 = -x^4 + 2ax^3 - a^2x^2 + c^2x^2 - 2ac^2x + a^2c^2$; whence $\overline{1 + t^2} \times x^4 - \overline{2a + 2dt} \times x^3 + \overline{a^2 - c^2 + d^2 - 2bct} \times x^2 + \overline{2ac^2 + 2bcd} \times x - a^2c^2 + b^2c^2 = 0$: from which the value of x may be found; and then, the value of $y\ (= \sqrt{c^2 - x^2})$ being known, the poſition of the points A and C, through which the line muſt paſs, will alſo become known.

If the given angle B be a right one, the point B will coincide with P; and therefore t in this caſe being $= 0$, the equation will become $x^4 - 2ax^3 + \overline{a^2 - c^2 + d^2} \times x^2 + \overline{2ac^2 + 2bcd} \times x - a^2c^2 + b^2c^2 = 0$.

When the circle touches the right-line AB, a will then be equal to b: and, in that caſe, the equation will be $\overline{1 + t^2} \times x^3 - \overline{2a + 2dt} \times x^2 + \overline{a^2 - c^2 + d^2 - 2act} \times x + 2ac^2 + 2acd = 0$, becauſe the two laſt terms $- a^2c^2 + b^2c^2$ deſtroying each other, the whole may, here, be divided by x.

Laſtly, if b be $= 0$, or the line AC, inſtead of touching a circle, be required to paſs through a given point, the equation will then become $\overline{1 + t^2} \times x^4 - \overline{2a + 2dt} \times x^3 + \overline{a^2 - c^2 + d^2} \times x^2 + 2ac^2x - a^2c^2 = 0$.

PROBLEM XXXI.

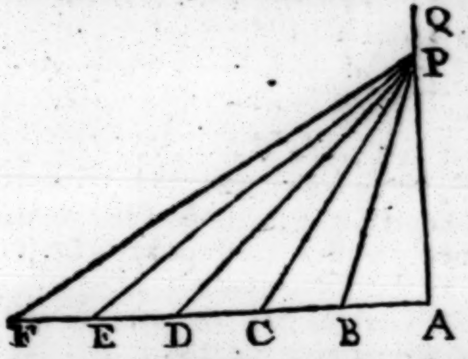
Suppoſing AQ *perpendicular to* AF, *and the given right-line* AF (50) *to be divided into five equal parts, in the points,* B, C, D, *and* E; *to find a point* P *in the perpendicular* AQ, *from which, if five right-lines be drawn to the points* B, C, D, E, *and* F, *the ſum of the outermoſt* PF + PE *ſhall be equal to the ſum of the three innermoſt* PD + PC + PB.

Put AP $= x$; then (*by Euc.* 47. 1.) BP $= \sqrt{100 + x^2}$, CP $= \sqrt{400 + x^2}$, &c. and conſequently, $\sqrt{100 + x^2} + \sqrt{400 + x^2} + \sqrt{900 + x^2} - \sqrt{1600 + x^2} -$

$\sqrt{2500+x^2}=0$. Now, by reflecting a little on the nature of the problem, it is eaſy to perceive that PF + PE muſt be greater than 90, ſeeing AF + AE is = 90; whence it appears that PD + PC + PB muſt alſo exceed 90, and that PC (conſidered as a mean between PD and PB) muſt be greater than 30: hence I conclude, that the value of AP, as it is ſomething leſs than PC, will be ſomewhere about 30; and therefore I write $30+e$ for x; and then, rejecting all the powers of e above the firſt, as inconſiderable, our equation ſtands thus, $\sqrt{1000+60e}+\sqrt{1300+60e}+\sqrt{1800+60e}-\sqrt{2500+60e}-\sqrt{3400+60e}=0$; which, by the method explained in *page* 174, will be transformed to

$$\sqrt{1000}+\frac{\sqrt{1000}\times 3e}{100}+\sqrt{1300}+\frac{\sqrt{1300}\times 3e}{130}+\sqrt{1800}+\frac{\sqrt{1800}\times 3e}{180}-50-6e-\sqrt{3400}-\frac{\sqrt{3400}\times 3e}{340}=0;$$

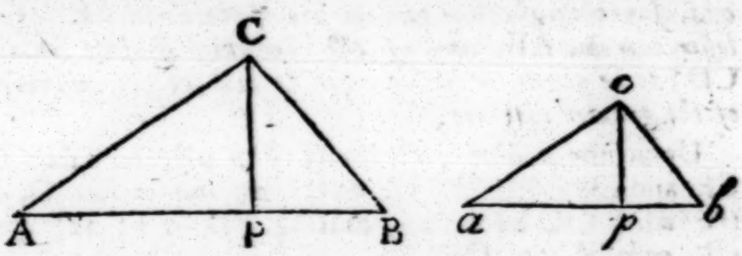
this, contracted, gives $1.8+1.37e=0$; whence $e=-1.3$, and conſequently $x=28.7$, nearly. Let, now, 28.7 be put $=x$; and then, by proceeding as above, we ſhall have $.0083+1.43e=0$; hence $e=-.0058$, and $x=28.6942$; which is true to the laſt figure.



PROBLEM XXXII.

The perimeter, AB + BC + AC, *and the perpendicular* CP *of a triangle* ABC *whoſe ſides are in harmonic proportion* (AB : BC :: AB — AC : AC — BC) *being given*; *to determine the triangle.*

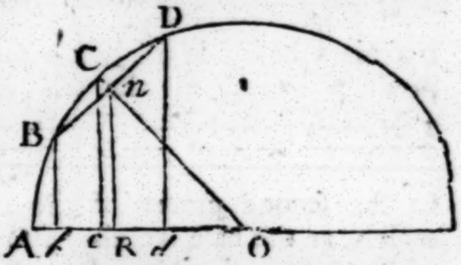
Let *abc* be another triangle, ſimilar to the propoſed one; and let $ab = 1$, $bc = x$, $ac = y$, $CP = a$, and $AB + BC + AC = b$: then, half the ſum of the three ſides of the triangle *abc* being $\frac{1 + x + y}{2}$, if from the ſame, each particular ſide be ſubtracted, and all the remainders be multiplied continually together, and that product, again, by the ſaid half ſum, we ſhall have $\frac{1 + x - y}{2} \times \frac{1 + y - x}{2} \times \frac{y + x - 1}{2} \times \frac{1 + x + y}{2}$, equal to the ſecond power of the area *abc* (*by prob.* 15): which, as the baſe is unity, alſo expreſſes $\frac{1}{4}$ of the ſquare of the perpendicular. But the ſquares of the ſides, as well as the ſides of ſimilar triangles, are proportional, &c. and therefore $\overline{1 + x + y}|^2 : b^2 :: \frac{\overline{1 + x - y} \times \overline{1 - x + y} \times \overline{y + x - 1} \times \overline{1 + x + y}}{4} : a^2$; whence we have $4a^2 \times \overline{1 + x + y} = \overline{1 + x - y} \times \overline{1 - x + y} \times \overline{y + x - 1} \times bb$: but the ſides AB, AC, and BC, being given in harmonic proportion, therefore 1, y, and x, muſt likewiſe be in the ſame proportion; that is, $1 : x :: 1 - y : y - x$; whence $y - x = x - xy$, and therefore $y = \frac{2x}{1 + x}$; which, ſubſtituted above, gives $\frac{4a^2 \times \overline{1 + 4x + x^2}}{1 + x} = \frac{\overline{1 + x^2}}{1 + x} \times \frac{1 + 2x - x^2}{1 + x} \times \frac{2x + x^2 - 1}{1 + x} \times b^2$, or $4a^2 \times \overline{1 + 4x + x^2} \times \overline{1 + x}|^2 = \overline{1 + x^2} \times \overline{1 + 2x - x^2} \times \overline{2x + x^2 - 1} \times b^2$; from which x will be found, and alſo



alſo $y\ (=\frac{2x}{1+x})$; and from thence the required ſides of the ſimilar figure ABC, will, by proportion, be likewiſe known.

PROBLEM XXXIII.

Let there be three equi-different arches, AB, AC, *and* AD; *and, ſuppoſing the ſine and co-ſine of the mean* AC, *of the leſſer extreme* AB, *and of the common difference* BC *(or* CD*) to be given, it is propoſed to find the ſine and co-ſine of the greater extreme* AD.



Upon the radius AO let fall the perpendiculars B*b*, C*c*, and D*d*; join B, D, and from the center O, let the radius OC be drawn, cutting BD in *n*: alſo draw *n*R parallel to C*c*, meeting AO in R: then, becauſe of the ſimilar triangles OC*c* and O*n*R, it will be, OC : O*n* : : C*c* : *n*R; and, OC : O*n* : : O*c* : OR: whence we have $nR=\frac{Cc\times On}{OC}$, and $OR=\frac{Oc\times On}{OC}$: but, ſince BC is equal to CD (and therefore B*n* equal to D*n*), *n*R will, it is plain, be an arithmetical mean between B*b* and D*d*, and ſo is equal to half their ſum, or $\frac{Bb+Dd}{2}$: and, for the very ſame reaſon, OR will be equal to $\frac{Ob+Od}{2}$; conſequently $\frac{Bb+Dd}{2}=\frac{Cc\times On}{OC}$, and $\frac{Ob+Od}{2}=\frac{Oc\times On}{OC}$; whence $Dd=\frac{Cc\times 2On}{OC}-Bb$ and $Od=\frac{Oc\times 2On}{OC}-Ob$; which, if the radius OC be ſuppoſed unity, will become $Dd=Cc\times 2On-Bb$; and $Od=Oc\times 2On-Ob$: from whence we have the two following theorems.

Theor. 1. *If the ſine of the mean, of any three equidifferent arches (the radius being ſuppoſed unity) be multiplied by twice the co-ſine of the common difference, and from the product, the ſine of either extreme be ſubtracted, the remainder will be the ſine of the other extreme.*

Theor. 2. *And if the co-ſine of the mean of three equidifferent arches be multiplied by twice the co-ſine of the common difference, and the co-ſine of either extreme be ſubtracted from the product, the remainder will be the co-ſine of the other extreme.*

PROBLEM XXXIV.

The ſine and co-ſine of an arch being given, to find the ſine and co-ſine of any multiple of that arch.

Let the given arch be repreſented by A, its ſine by x, and co-ſine by $\frac{1}{2}y$, the radius being unity. Then, ſince the arch A may be conſidered as an arithmetical-mean between 0 and 2A, we ſhall, by the firſt of the two preceding theorems, have

ſine of 2A (= ſine of A $\times y$ — ſine of 0) $= xy$;
ſine of 3A (= ſine of 2A $\times y$ — ſine of A) $= xy^2 - x$;
ſine of 4A (= ſine of 3A $\times y$ — ſine of 2A $= xy^3 - xy - xy) = xy^3 - 2xy$;
ſine of 5A (= ſine of 4A $\times y$ — ſine of 3A $= xy^4 - 2xy^2 - xy^2 + x) = xy^4 - 3xy^2 + x$;
ſine of 6A (= ſine of 5A $\times y$ — ſine of 4A $= xy^5 - 3xy^3 + xy - xy^3 + 2xy) = xy^5 - 4xy^3 + 3xy$;
ſine of 7A (= ſine of 6A $\times y$ — ſine of 5A $= xy^6 - 4xy^4$ $3xy^2 - xy^4 + 3xy^2 - x) = xy^6 - 5xy^4 + 6xy^2 - x$:

whence, *univerſally*, the ſine of the multiple-arch nA, where n denotes any whole poſitive number, whatever, will be truly expreſſed by $x \times$ into this ſeries

$$y^{n-1} - \frac{n-2}{1} \times y^{n-3} + \frac{n-3}{1} \times \frac{n-4}{2} \times y^{n-5} - \frac{n-4}{1} \times \frac{n-5}{2} \times \frac{n-6}{3} \times y^{n-7}$$

&c. Moreover, from the ſecond theorem, we have,
Co-ſine of 2A (= co-ſine of A $\times y$ — co-ſine of 0 $= \frac{y^2}{2} - 1) = \frac{y^2 - 2}{2}$;

Co-ſine of 3A (= co-ſine of 2A × y — co-ſine of A = $\frac{y^3}{2} - y - \frac{y}{2}$) = $\frac{y^3 - 3y}{2}$;

Co-ſine of 4A (= co-ſine of 3A × y — co-ſine of 2A = $\frac{y^4 - 3y^2}{2} - \frac{y^2 - 2}{2}$) = $\frac{y^4 - 4y^2 + 2}{2}$;

whence, *univerſally*, the co-ſine of the multiple-arch nA will be truly repreſented by $\frac{y^n}{2} - \frac{ny^{n-2}}{2} + \frac{n}{2} \times \frac{n-3}{2} \times y^{n-4} - \frac{n}{2} \times \frac{n-4}{2} \times \frac{n-5}{3} \times y^{n-6} + \frac{n}{2} \times \frac{n-5}{2} \times \frac{n-6}{3} \times \frac{n-7}{4} \times y^{n-8}$ *&c.* which ſeries, as well as that for the ſine, is to be continued till the indices of y become nothing, or negative.

But, if you would have the ſine expreſſed in terms of x *only*, then, becauſe the ſquare of the ſine + the ſquare of the co-ſine is always equal to the ſquare of the radius, and therefore, in this caſe, $x^2 + \frac{1}{4}y^2 = 1$, it is manifeſt that the ſines of all the odd multiples of the given arch A, wherein only the even powers of y enter, may be exhibited in terms of x only, without ſurd quantities: ſo that $4 - 4x^2$ being ſubſtituted for its equal y^2, in the ſines of the aforementioned arches, we ſhall have

1ſt. Sine of 3A = $3x - 4x^3$;
2d. Sine of 5A = $5x - 20x^3 + 16x^5$;
3d. Sine of 7A = $7x - 56x^3 + 112x^5 - 64x^7$;
4th. Sine of 9A = $9x - 120x^3 + 432x^5 - 576x^7 + 256x^9$;
&c. *&c.*

And, *generally*, if the multiple-arch be denoted by nA, then the ſine thereof will be truly repreſented by $nx - \frac{n}{1} \times \frac{n^2-1}{2 \cdot 3} \times x^3 + \frac{n}{1} \times \frac{n^2-1}{2 \cdot 3} \times \frac{n^2-9}{4 \cdot 5} \times x^5 - \frac{n}{1} \times \frac{n^2-1}{2 \cdot 3} \times \frac{n^2-9}{3 \cdot 4} \times \frac{n^2-25}{5 \cdot 6} \times x^7 + \frac{n}{1} \times \frac{n^2-1}{2 \cdot 3} \times \frac{n^2-9}{4 \cdot 5} \times \frac{n^2-25}{6 \cdot 7} \times \frac{n^2-49}{8 \cdot 9} \times x^9$ *&c.*

From this ſeries the ſine of the ſub-multiple of any arch, where the number of parts is odd, may alſo be found, ſuppoſing (s) the ſine of the whole arch to be given: for let x be the required ſign of the ſub-multiple, and n the number of equal parts into which the whole arch is divided; then, by what has been already ſhewn, we ſhall have $nx - \frac{n}{1} \times \frac{n^2-1}{2 \cdot 3} \times x^3 + \frac{n}{1} \times \frac{n^2-1}{2 \cdot 3} \times \frac{n^2-9}{4 \cdot 5} \times x^5$ &c. $= s$: from the ſolution of which equation the value of x will be known. Hence alſo, we have an equation for finding the ſide of a regular polygon inſcribed in a circle: for, ſeeing the ſine of any arch is equal to half the chord of double that arch, let $\frac{1}{2}v$ and $\frac{1}{2}w$ be wrote above for x and s reſpectively, and then our equation will become $\frac{nv}{2} - \frac{n}{1} \times \frac{n^2-1}{2 \cdot 3} \times \frac{v^3}{8} + \frac{n}{1} \times \frac{n^2-1}{2 \cdot 3} \times \frac{n^2-9}{4 \cdot 5} \times \frac{v^5}{32}$ &c. $= \frac{1}{2}w$, or $nv - \frac{n}{1} \times \frac{n^2-1}{2 \cdot 3} \times \frac{v^3}{4} + \frac{n}{1} \times \frac{n^2-1}{2 \cdot 3} \times \frac{n^2-9}{4 \cdot 5} \times \frac{v^5}{16}$ &c. $= w$; expreſſing the relation of chords, whoſe correſponding arches are in the ratio of 1 to n. But, when the greater of the two arches becomes equal to the whole periphery, its chord (w) will be nothing, and then the equation, by dividing the whole by nv, will be reduced to $1 - \frac{n^2-1}{2 \cdot 3} \times \frac{v^2}{4} + \frac{n^2-1}{2 \cdot 3} \times \frac{n^2-9}{4 \cdot 5} \times \frac{v^4}{16} - \frac{n^2-1}{2 \cdot 3} \times \frac{n^2-9}{4 \cdot 5} \times \frac{n^2-25}{5 \cdot 6} \times \frac{v^6}{64}$ &c. $= 0$; where n is the number of ſides, and v the ſide of the polygon.

From the foregoing ſeries, that given by Sir *Iſaac Newton*, in *Phil. Tranſ.* mentioned in *p.* 242 of this Treatiſe, may alſo be eaſily derived. For, if the arch A and its ſine x be taken indefinitely ſmall, they will be to one another in the ratio of equality, indefinitely near, by what has been proved at *p.* 246; in which caſe, the general expreſſion, by writing A inſtead of x, will become $nA - \frac{n}{1} \times \frac{n^2-1}{2 \cdot 3} \times A^3 + \frac{n}{1} \times \frac{n^2-1}{2 \cdot 3} \times \frac{n^2-9}{4 \cdot 5} \times A^5 - \frac{n}{1} \times \frac{n^2-1}{2 \cdot 3} \times \frac{n^2-9}{4 \cdot 5} \times \frac{n^2-25}{6 \cdot 7} \times A^7$ &c.

Therefore, if n be now ſuppoſed indefinitely great, ſo that the multiple-arch nA may be equal to any given arch z, the ſquares of the odd numbers, 1, 3, 5, &c. in the factors n^2-1, n^2-9, n^2-25, &c. may be rejected as nothing, or inconſiderable, in reſpect of n^2; and then the foregoing ſeries will become $nA - \frac{n^3A^3}{2 \cdot 3} + \frac{n^5A^5}{2 \cdot 3 \cdot 4 \cdot 5} - \frac{n^7A^7}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7}$ &c. wherein, if for nA, its equal z, be ſubſtituted, we ſhall then have $z - \frac{z^3}{2 \cdot 3} + \frac{z^5}{2 \cdot 3 \cdot 4 \cdot 5} - \frac{z^7}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7}$ &c. which is the ſine of the arch z, and the ſame with *that* before given.

Moreover the aforegoing general expreſſions may be applied, with advantage, in the ſolution of cubic, and certain other higher equations, included in this form, *viz.* $z^n - az^{n-2} + \frac{n-3}{2n} \times a^2z^{n-4} - \frac{n-4}{2n} \times \frac{n-5}{3n} \times a^3z^{n-6} + \frac{n-5}{2n} \times \frac{n-6}{3n} \times \frac{n-7}{4n} \times a^4z^{n-8}$ &c. $= f$.

For, if z be put $= y\sqrt{\frac{a}{n}}$, the equation will be tranſformed to $\overline{\frac{a}{n}}\Big|^{\frac{n}{2}} \times$ into this ſeries $y^n - ny^{n-2} + \frac{n}{1} \times \frac{n-3}{2} \times y^{n-4} + \frac{n}{1} \times \frac{n-4}{2} \times \frac{n-5}{3} \times y^{n-6}$ &c. $= f$, and conſequently $\frac{y^n}{2} - \frac{n}{2} \times y^{n-2} + \frac{n}{2} \times \frac{n-3}{2}$ $\times$

$$\times y^{n-4} - \frac{n}{2} \times \frac{n-4}{2} \times \frac{n-5}{3} \times y^{n-6} \text{ \&c.} = \frac{\frac{1}{2}f}{\left.\frac{a}{n}\right|^{\frac{n}{2}}}:$$ from

whence, as it is proved above, that the former part of the equation (and therefore its equal) repreſents the co-ſine of n times the arch whoſe co-ſine is $\frac{1}{2}y$, we have the following rule.

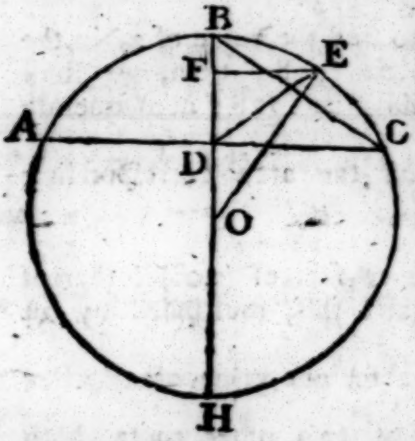
Find, from the tables, the arch whoſe natural co-ſine is $\frac{\frac{1}{2}f}{\left.\frac{a}{n}\right|^{\frac{n}{2}}}$ *(or its log. co ſine = log.* $\frac{1}{2}f - \frac{n}{2}$ *log.* $\frac{a}{n}$*) the radius being unity; take the* nth *part of that arch, and find its co-ſine, which multiply by* $2\sqrt{\frac{a}{n}}$*, and the product will be the true value of* z*, in the propoſed equation* $z^n - az^{n-2} + \frac{n-3}{2n} \times a^2z^{n-4} - \frac{n-4}{2n} \times \frac{n-5}{3n} \times a^3z^{n-6}$ *&c.*

Thus, let it be required to find the value of z, in the cubic equation $z^3 - 432z = 1728$; then, we ſhall have $n = 3$, $a = 432$, and $f = 1728$; conſequently $\frac{\frac{1}{2}f}{\left.\frac{a}{n}\right|^{\frac{n}{2}}}$ $\left(= \frac{864}{\left.144\right|^{\frac{3}{2}}}\right) = .5$, and the arch correſponding thereto $= 60°$; whence the co-ſine of $(20°)$ $\frac{1}{3}$ thereof will be found .9396926; and this, multiplied by 24 $\left(= 2\sqrt{\frac{a}{n}}\right)$ gives 22,55262 for one value of z. But beſides this, the equation has two other roots, both of which may be found after the very ſame manner: for, ſince 0,5 is not only the co-ſine of 60°, but alſo of $60° + 360°$, and $60° + 2 \times 360°$, let the co-ſine of $(140°)$ $\frac{1}{3}$ of the former of theſe arches be now taken; which is −,7660444, and muſt be expreſſed with a negative ſign, becauſe the arch correſponding is greater than one right-angle, and leſs than three. Then, the value thus found being, in like manner, multiplied by 24 $\left(= 2\sqrt{\frac{a}{n}}\right)$, we ſhall thence get — 18,38506 for another of the roots: whence the third, or remaining root will alſo be known; for, ſeeing the equation wants the ſecond term, the poſitive and negative roots do here mutually deſtroy each other; and therefore the remaining root muſt be — 4,16756, the difference of the two former, with a negative ſign.

PROBLEM XXXV.

From a given circle ABCH *it is propoſed to cut off a ſegment* ABC, *ſuch, that a right-line* DE *drawn from the middle of the chord,* AC, *to make a given angle therewith, ſhall divide the arch* BC *of the ſemi-ſegment into two equal parts* BE *and* EC.

Let the chord BC be drawn, and upon the diameter HDB let fall the perpendicular EF: put the radius OB of the circle $= r$, and the tangent of the given angle CDE (anſwering to that radius) $= t$, and let OF $= z$; then will EF $= \sqrt{rr - zz}$, and BC $(= 2\text{EF}) = 2\sqrt{rr - zz}$, and conſequently BD $\left(= \frac{\text{BC}^2}{\text{BH}}\right) = \frac{4r^2 - 4z^2}{2r} = \frac{2 \cdot \overline{r+z} \cdot \overline{r-z}}{r}$: from which taking BF $= r - z$, we have DF $= \frac{\overline{r+2z} \times \overline{r-z}}{r}$.



But, by trigonometry, EF : DF : : rad. : tang. DEF, that is, $\sqrt{rr - zz} : \frac{\overline{r+2z} \cdot \overline{r-z}}{r} :: r : t$. Whence we have $\overline{r+2z}|^2 \times \overline{r-z}|^2 = t^2 \times \overline{r^2 - z^2}$; where the whole being divided by $r - z$, their reſults $\overline{r + 2z}|^2 \times \overline{r - z} = t^2 \times \overline{r + z}$: which, ordered, gives

$$z^3 - \frac{3rr - tt}{4} \times z = \frac{r}{4} \times \overline{rr - tt}.$$

Put $\frac{3rr - tt}{4} = a$, and $\frac{r}{4} \times \overline{rr - tt} = f$; then it will be $z^3 - az = f$. Therefore find, from the tables, the arch whoſe co-ſine is $\frac{\frac{1}{2}f}{\frac{1}{3}a\sqrt{\frac{1}{3}a}}$ (the radius being unity); take $\frac{1}{3}$ thereof, and find its coſine; which, multiplied by $2\sqrt{\frac{1}{3}a}$, gives the true value of z (*ſee the laſt problem.*)

Thus, for example, let the radius OB ($= r$) $= 1$, and the given angle CDE $= 25°$, whoſe tangent (t) is therefore $= .4663$; whence $\frac{1}{3}a = .23188$, and $\frac{1}{2}f = .09782$.

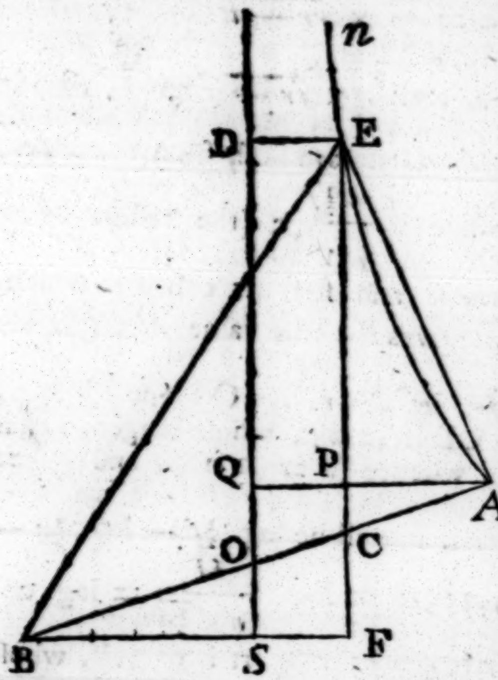
Now, *by logarithms*, it will be log. $\frac{1}{2}f$ — log. $\frac{1}{3}a$ — $\frac{1}{2}$ log. $\frac{1}{3}a = -1.9425328 =$ log. $\frac{\frac{1}{2}f}{\frac{1}{3}a\sqrt{\frac{1}{3}a}} =$ log. co-ſine of $28° 50'$; whereof the third part is $9° 36\frac{2}{3}'$, whoſe log. co-ſine (to the radius 1) is -1.9938609; which added to the $\frac{1}{2}$ log. of $\frac{1}{3}a$ ($= -1.6826316$) gives $-1.6764925 =$ log. of 0.47478, whoſe double $.94956$, is the true value of z, or FO: whence the correſponding arch BE $= 18° 16\frac{1}{2}'$, and conſequently BC ($= 2$BE) $= 36° 33'$.—By means of this problem that portion of a ſpherical ſurface repreſenting the apparent figure of the ſky is determined.

PROBLEM XXXVI.

The baſe AB, *and the difference of the angles at the baſe being given, while the angles themſelves vary; to find the* locus *of the vertex* E *of the triangle.*

Let the baſe AB be biſected in O, and the angle BOD ſo conſtituted as to exceed its ſupplement AOD by the given difference of EAB and EBA; and let ED, APQ, BSF be perpendicular, and EPF parallel to OD: then, ſince the angle BCE (BOD) as much exceeds

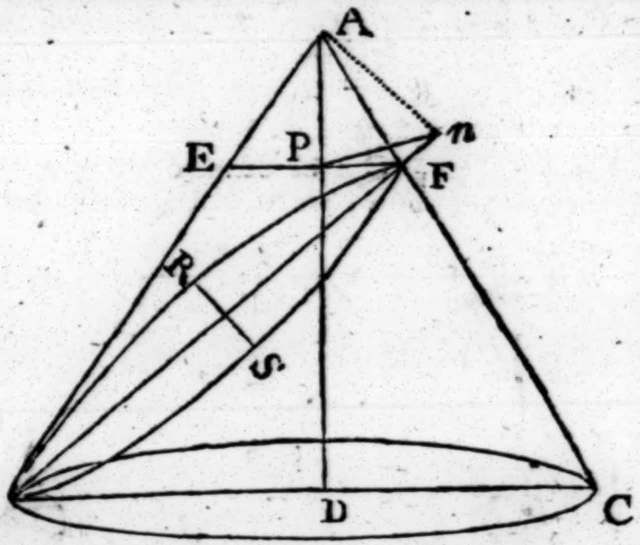
ACE, as CAE exceeds CBE, it is evident that the ſum of the two angles BCE, CBE, of the triangle BCE, is equal to the ſum of the two angles ACE, CAE of the triangle ACE; and, conſequently, that the remaining angles AEC and BEC, are equal the one to the other: therefore, by reaſon of the ſimilar triangles EFB, EAP, we have $EF : EP :: BF : AP$, that is, $OD + OQ : OD - OQ :: QA + DE : QA - DE$; whence, by compoſition and diviſion, $2OD : 2OQ :: 2QA : 2DE$; wherefore $OD \times DE$ is $= OQ \times QA$; which is the known property of an equilateral hyperbola with reſpect to its aſymptote.



PROBLEM XXXVII.

To find the ſolidity of a conical ungula BFCB, *cut off by a plane* BRFSB *paſſing through one extremity of the baſe-diameter.*

Let EPF be parallel to the baſe-diameter BC, cutting AD the axis of the cone in P; alſo, let A*n* be perpendicular to BF; join P, *n*, and let RS be the conjugate axis of the elliptical ſection BRFSB: then the part ABF, above the ſaid ſection, being an oblique elliptical-cone, its ſolidity will be expreſſed by $.7854 \times SR \times BF \times \frac{An}{3}$, that is, by the area of its baſe BRFSB drawn into $\frac{1}{3}$ of the perpendicular height. But the triangles BCF and AP*n* will appear to be equi-angular; for, APF and A*n*F being both right-angles, the circumference of a circle, deſcribed on the diameter AF, will paſs through P and *n*; and ſo the angles AF*n* (BFC) and AP*n*, as well as AFP (FCB) and A*n*P,



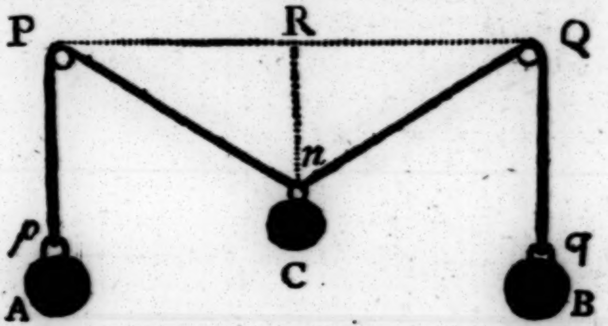
inſiſting on the ſame arch, are, reſpectively, equal. Hence we have BC : BF :: A*n* : AP; and therefore BF × A*n* = BC × AP: this value being ſubſtituted above, the content of the part ABF becomes SR × BC × AP × .2618: which, becauſe SR is known to be $= \sqrt{BC \times EF}$, is farther reduced to $BC \times AP \times \sqrt{BC \times EF} \times .2618$. This ſubtracted from, $BC^2 \times AD \times .2618$, the content of the whole cone ABC, leaves

$\overline{BC^2 \times AD - BC \times AP \times \sqrt{BC \times EF}} \times .2618$ for the required ſolidity of the *ungula* BCF; which, becauſe $AD = \frac{DP \times BC}{BC - EF}$, and $AP = \frac{DP \times EF}{BC - EF}$, will be reduced to $\frac{BC^2 - EF \times \sqrt{BC \times EF}}{BC - EF} \times .2618DP \times BC$.

PROBLEM XXXVIII.

Let A *and* B *be two equal weights, made faſt to the ends of a thread, or perfectly flexible line* p*P*n*Q*q*, ſupported by two pins, or tacks,* P, Q, *in the ſame horizontal plane; over which pins the line can freely ſlide either way; and let* C *be another weight, faſtened to the thread, in the middle, between* P *and* Q: *now the queſtion is, to find the poſition of the weight* C, *or its diſtance below the horizontal line* PQ, *to retain the other two weights* A *and* B *in equilibrio.*

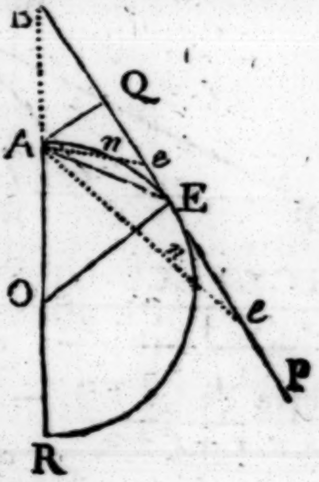
Let PR (= ½PQ) be denoted by a, and Rn (the diſtance ſought) by x; and then Pn, or Qn, will be repreſented by $\sqrt{a^2 + x^2}$. Therefore, *by the reſolution of forces*, it will be, as $\sqrt{a^2 + x^2}$ (Pn) : x (Rn) : : the whole force of the weight A in the direction Pn, to $\frac{Ax}{\sqrt{a^2+x^2}}$ its force in the direction nR, whereby it endeavours to raiſe the weight C; which quantity alſo expreſſes the force of the weight B in the ſame direction: but the ſum of theſe two forces, ſince the weights are ſuppoſed to reſt in equilibrio, muſt be equal to that of the weight C; that is, $\frac{2Ax}{\sqrt{a^2 + x^2}} = C$; whence we have $4A^2x^2 = C^2a^2 + C^2x^2$, and conſequently $x = \frac{aC}{\sqrt{4A^2 - C^2}}$.



PROBLEM XXXIX.

To determine the poſition of an inclined plane AE, *along which a heavy body deſcending by the force of its own gravity from a given point* A, *ſhall reach a right-line* BP, *given by poſition, in the leaſt time poſſible.*

Through the given point A, perpendicular to the horizon, let there be drawn the right-line RB, meeting BP in B; alſo conceive the ſemi-circle AER to be deſcribed, touching BP in E; then let AE be drawn, which will be the poſition required; becauſe the time of deſcent along the chord AE being equal to *that* along any other chord A*n*, it will conſequently be leſs than the time of the deſcent along A*e*, whereof A*n* is only a part: therefore, if AQ and OE be now made perpendicular to BP, we ſhall have (by reaſon of the ſimilar triangles) $AB : AQ :: AB + AO : (OE)\ AO$; whence, by multiplying extremes and means, $AB \times AO = AQ \times AB + AQ \times AO$; therefore $AB \times AO - AQ \times AO = AQ \times AB$, and $AO\ (OE) = \dfrac{AQ \times AB}{AB - AQ}$; from which BE and AE are alſo given.

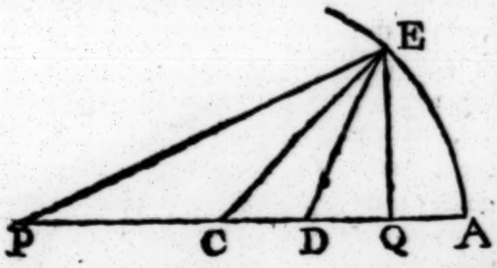


The geometrical conſtruction of this problem is extremely eaſy; for, if AQ (as above) be drawn perpendicular to BP, and the angle OAQ be biſected by AE, the thing is done: becauſe, OE being drawn parallel to AQ, the angle OEA is = QAE = EAO; and ſo, AO being = OE, the ſemi-circle that touches BP, will paſs through A.

PROBLEM XL.

A ray of light, from a lucid point P *in the axis* AP *of a concave ſpherical ſurface, is reflected at a given point* E *in that ſurface; to find the point* D *where the reflected ray meets the axis.*

Draw EQ perpendicular to AP, and from the center C let CE be drawn; alſo make CE $= a$, CQ $= b$, CP $= c$, CD $= x$; and (*by Euc.* 12. 3.) PE will be $= \sqrt{a^2 + c^2 + 2bc}$: wherefore, the angles of incidence and reflection, CEP and CED being equal, we have, as

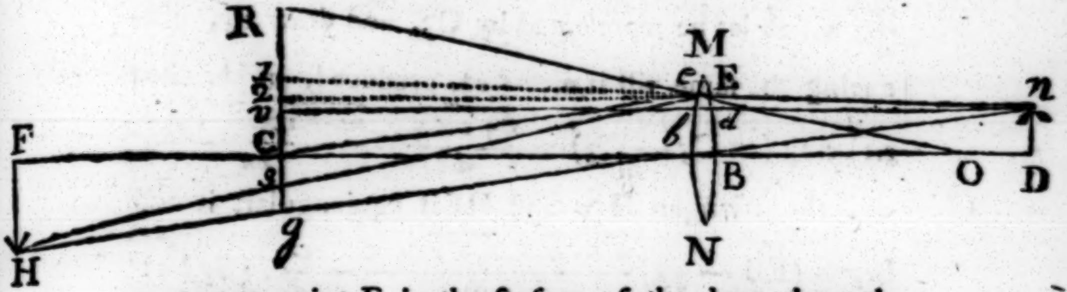


PC (c) : CD (x) : : PE $(\sqrt{a^2 + c^2 + 2bc})$: ED $= \frac{x\sqrt{a^2 + c^2 + 2bc}}{c}$; alſo, for the ſame reaſon, we have PE × ED — PC × CD = EC², that is, $\frac{x \times \overline{a^2 + c^2 + 2bc}}{c} - cx = a^2$; which, reduced, gives $x = \frac{ca^2}{a^2 + 2bc}$, ſhewing how far from the center the ray cuts the axis. But if the lucid point P be ſuppoſed infinitely remote, ſo that the ray PE may be conſidered as parallel to the axis AP, the expreſſion will be more ſimple; for then a^2, in the diviſor, may be rejected as nothing in compariſon of $2bc$; that being done, CD or x becomes $= \frac{a^2}{2b}$; which, therefore, if E be taken near the vertex A, will be $= \frac{1}{2}a$, very nearly.

PROBLEM XLI.

To find the magnitude and poſition of an image formed by refraction at a given lens.

Let MN be the given lens, DOBCF the axis thereof, and Dn the object whoſe image FH we would find; alſo let CB be the radius of that ſurface of the lens MBN, which is neareſt the object, and Ob that of the other ſurface: make RCg perpendicular to DF, and from



n, to any point E in the ſurface of the lens, draw the incident ray nE, and let the continuation thereof be E1, and let the direction of the ſame ray, after the firſt refraction at E, be E2; and, after it is refracted a ſecond time, at e, let its direction be e3H; draw CE and OeR, and make ndv parallel to DF, calling Ob, b; CB, c; BD, d; Dn, p; and the diſtance of the point E from the axis DF, x; and let the ſine of incidence be to the ſine of refraction, out of air into glaſs, as m to n. Then, the thickneſs of the lens being looked upon as inconſiderable in reſpect of the focal diſtance Fb, we ſhall have, as $d : x - p$ (Ed) $: : d + c$ (nv) $:$ $\frac{dx + cx - dp - cp}{d} = v1$; which added to C$v$ (p) gives $\frac{dx + cx - cp}{d} =$ C1: therefore $m : n : : \frac{dx + cx - cp}{d} :$ $\frac{n \times \overline{dx + cx - cp}}{dm} =$ C2. Moreover, b (Ob) $: x$ (BE) $: :$ $b + c$ (OC) $: \frac{\overline{b + c} \times x}{b} =$ CR; whence R2 (= CR − C2) $= \frac{\overline{b + c} \times x}{b} - n \times \frac{dx + cx - cp}{dm}$: but $n : m : :$ R2 : R3 $= \frac{mx \times \overline{b + c}}{bn} + \frac{cp - cx - dx}{d}$; and therefore

C3 (=R3—RC) $= \frac{\overline{m-n} \times x \times \overline{b+c}}{bn} + \frac{cp-cx-dx}{d}$.
Let x be now taken = 0, and then C3 will become $\frac{cp}{d}$, which let be repreſented by C3, and draw Bg, producing the ſame till it meets $e3$, produced, in H: then $g3$ being (= Cg — C3) $= \frac{\overline{d+c} \times x}{d} - \frac{\overline{m-n} \times \overline{b+c} \times x}{bn}$, and the triangles H3$g$ and HEB equiangular, it will be, as (EB — $g3$) $\frac{\overline{m-n} \times \overline{b+c} \times x}{bn} - \frac{cx}{d}$: (Bg) c :: (EB) x : BH $= \frac{nbcd}{\overline{m-n} \times \overline{b+c} \times d - nbc}$ = the required diſtance of the image from the lens; and as c (BC) : $\frac{cp}{d}$ (Cg) :: BH (or BF) : FH $= \frac{\text{BF} \times p}{\text{BD}} =$ $\frac{nbcp}{\overline{m-n} \times \overline{b+c} \times d - nbc}$ (= HF) the magnitude of the image, or its linear amplification.

COROL. 1.

Becauſe the values of BH and HF are alike affected by b and c, it follows that both the diſtance and magnitude of the image will remain unaltered, if the place of the lens be the ſame, let which ſide you will be turned towards the object.

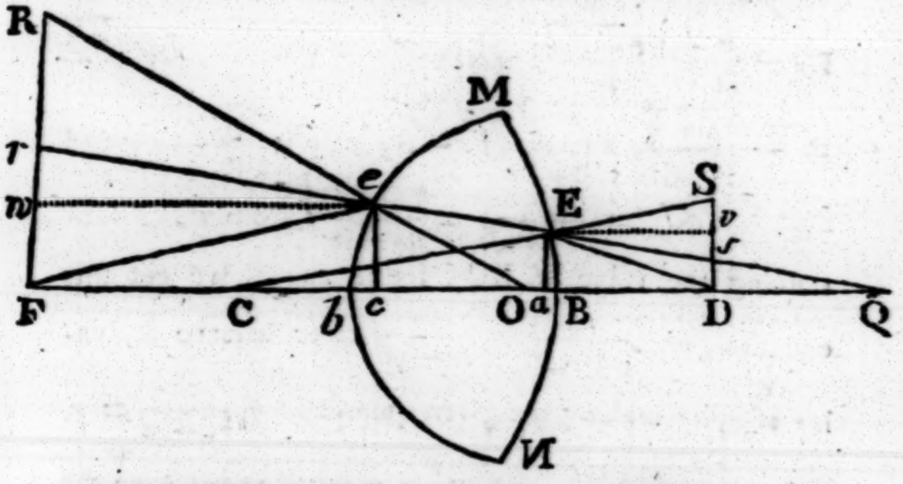
COROL. 2.

If d be made infinite, or the diſtance of the object from the lens be ſuppoſed indefinitely great, BF will become $\frac{nbc}{\overline{m-n} \times \overline{b+c}}$; which is the principal focal diſtance, at which the parallel rays unite, and this diſtance, when both ſides of the lens have the ſame convexity, or b is $= c$, will become $= \frac{nb}{2m-2n}$: but in a *plano-convex*, where b is infinite, it will be $= \frac{nc}{m-n}$;

and, in a *meniſcus*, where b is negative, or one ſurface concave and the other convex, it will be $\frac{nbc}{\overline{m-n}\times\overline{b-c}}$.

The ſame anſwered otherwiſe, allowing alſo for the thickneſs of the lens.

Suppoſing, as before, that F is the place of the image of an object at D, let FR and DS be ſuppoſed perpendicular to the axis FBQ, interſecting the continuation of Ee (the intercepted part of the ray $DEeF$) in r and s, and meeting the radii Oe, EE (produced) in R and S; likewiſe let Ea and ec be perpendicular to QF, and Ev and ew parallel thereto: then, becauſe the ray is ſuppoſed to be indefinitely near the axis, ac may be taken for the thickneſs of the lens, which let be denoted by t; putting $bF = z$, $ce = y$, $aE = x$, $Ob = b$, $CB = c$, and $BD = d$ *(as before)*. By ſimilar triangles, $Ca\ (c) : aE\ (x) :: CD\ (c+d) : DS = \frac{x\times\overline{c+d}}{c}$; and, by the law of refraction, $m : n :: DS : Ss = \frac{n}{m}\times\frac{x\times\overline{c+d}}{c}$: whence $Ds\ (= DS - Ss) = \overline{1-\frac{n}{m}}\times\frac{x\times\overline{c+d}}{c}$, and $vs\ (= aE - Ds) =$



$x\times\overline{r-\frac{qd}{c}}$, by making $r=\frac{n}{m}$, and $q=1-r$.

Now, *vs* : E*v* (BD) : : *a*E : *a*Q (BQ;) that is, $x\times\overline{r-\frac{qd}{c}} : d :: x : \frac{cd}{cr-qd}=$ BQ; which is given from hence.

Again, in the very ſame manner, O*c* (*b*) : *ce* (*y*) : : OF (*b* + *z*) : FR $=\frac{y\times\overline{b+z}}{b}$; and $m : n ::$ FR : R*r* $=\frac{n}{m}\times\frac{y\times\overline{b+z}}{b}$: whence F*r* $=\overline{1-\frac{n}{m}}\times\frac{y\times\overline{b+z}}{b}$ $=\frac{qy\times\overline{b+z}}{b}$, and *wr* (F*r*—*ce*) $=y\times\overline{\frac{qz}{b}-r}$; and therefore *c*Q $(=\frac{we\times ce}{wr})=\frac{bz}{qz-br}$: from which ſubtracting the value of BQ, found above, we get this equation, *viz.* $\frac{bz}{qz-br}-\frac{cd}{cr-qd}=t$: whence the value of *z*, by making the given quantity $t+\frac{cd}{rc-qd}=g$, comes out $=\frac{rbg}{qg-b}$. But, if you had rather have the ſame in original terms, it is but ſubſtituting for *g*; whence, after reduction, $z=\frac{rbcd+rbt\times\overline{rc-qd}}{qd\times\overline{b+c}-rbc+qt\times\overline{rc-qd}}$: which, by reſtoring *m* and *n*, becomes

$$z=\frac{mnbcd+nbt\times\overline{nc-\overline{m-n}\times d}}{\overline{m-n}\times md\times\overline{b+c}-mnbc+\overline{m-n}\times t\times\overline{nc-\overline{m-n}\times d}};$$

where, if *t* be taken = 0, we ſhall have

$$z=\frac{nbcd}{\overline{m-n}\times d\times\overline{b+c}-nbc},$$

the very ſame as was found by the preceding method.

APPENDIX:

CONTAINING THE

CONSTRUCTION

OF

GEOMETRICAL PROBLEMS,

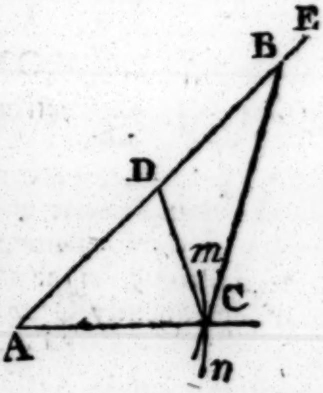
WITH

The Manner of reſolving the ſame numerically.

PROBLEM I.

The baſe, the ſum of the two ſides, and the angle at the vertex of any plane triangle being given, to deſcribe the triangle,

CONSTRUCTION.



DRAW the indefinite right-line AE, in which take AB equal to the ſum of the ſides, and make the angle ABC equal to half the given angle at the vertex, and upon the point A, as a center, with a radius equal to the given baſe, let a circle *n*C*m* be deſcribed, cutting BC in C; join A, C, and make the angle BCD = CBD, and let CD cut AB in D; then will ACD be the triangle that was to be conſtructed.

DEMONSTRATION.

Becauſe the angles BCD and CBD are equal, therefore is CD = DB (*Euc.* 6. 1.) and conſequently AD + DC = AB: likewiſe, for the ſame reaſon, the angle ADC (= BCD + CBD, *Euc.* 32. 1.) is equal to 2CBD. Q. E. D.

Method of calculation.

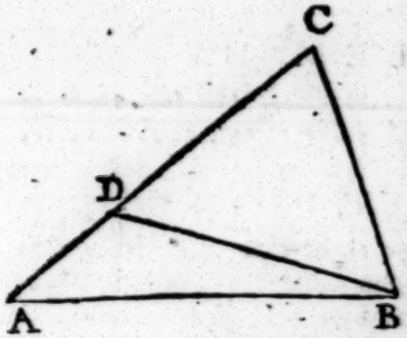
In the triangle ABC are given the two ſides AB, AC, and the angle ABC, whence the angle A is known; then in the triangle ADC will be given all the angles, and the baſe AC; whence the ſides AD and DC will alſo be known.

PROBLEM II.

The angle at the vertex, the baſe, and the difference of the ſides being given, to determine the triangle.

CONSTRUCTION.

Draw AC at pleaſure, in which take AD equal to the difference of the ſides, and make the angle CDB equal to the complement of half the given angle to a right angle; then from the point A draw AB equal to the given baſe, ſo as to meet DB in B, and make the angle DBC = CDB, then will ABC be the triangle required.



DEMONSTRATION.

Since (*by conſtruction*) the angles CDB and DBC are equal, CB is equal to CD, and therefore CA — CB = AD: moreover, each of thoſe equal angles being equal to the complement of half the given angle, their ſum, which is the ſupplement of the angle C, muſt therefore be equal to two right angles — the (whole) given angle, and conſequently C = the given angle. Q. E. D.

Method of calculation.

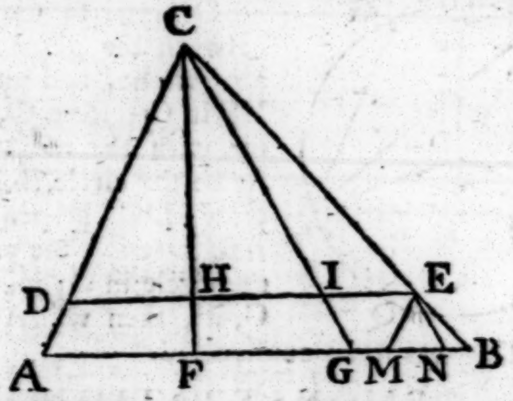
In the triangle ABD are given the ſides AB, AD, and the angle ADB, whence the angle A will be given, and conſequently BC and AC.

PROBLEM III.

The angle at the vertex, the ratio of the including ſides, and either the baſe, the perpendicular, or difference of the ſegments of the baſe being given, to deſcribe the triangle.

CONSTRUCTION.

Draw CA at pleaſure, and make the angle ACB equal to the angle given; take CB to CA in the given ratio of the ſides, and join A, B: then, if the baſe be given, let AM be taken equal thereto, and draw ME parallel to CA meeting CB in E, and make ED parallel to AB; but if the perpendicular be given, let fall CF perpendicular to AB, in which take CH equal to the given perpendicular, and draw DHE parallel to AB; laſtly, if the difference of the ſegments of the baſe be given, take FG = AF, and join, C, G, and take GN equal to the difference of the ſegments given, drawing NE parallel to CG, and ED to BA (as before;) then will CDE be the triangle which was to be conſtructed.



DEMONSTRATION.

Becauſe of the parallel lines AB, DE; ME, AC; and NE, GC; thence is DE = AM, and EI = NG; and alſo CD : CE : : CA : CB (*Euc.* 4. 6.) Q.E.D.

Method of calculation.

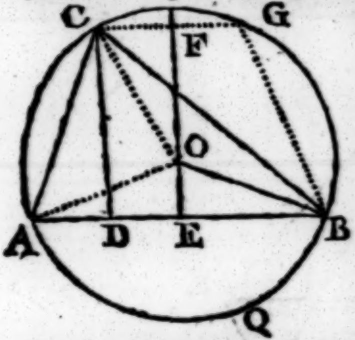
Let AC be aſſumed at pleaſure; then, the ratio of AC to BC being given, BC will become known; and therefore in the triangle ACB will be given two ſides and the included angle, whence the angles B and A, or E and D will be found; then in the triangle EDC, EHC, or EIC, according as the baſe, perpendicular, or the difference of the ſegments of the baſe is given, you will have one ſide and all the angles, whence the other ſides will be known.

PROBLEM IV.

The angle at the vertex, and the ſegments of the baſe, made by a perpendicular falling from the ſaid angle, being given, to deſcribe the triangle.

CONSTRUCTION.

Let the given ſegments of the baſe be AD and DB; biſect AB by the perpendicular EF, and make the angle EBO equal to the difference between the given angle and a right one, and let BO meet EF in O; from O, as a center, with the radius OB, deſcribe the circle BGAQ, and draw DC perpendicular to AB, meeting the periphery of the circle in C; join A, C and C, B, then will ACB be the triangle that was to be conſtructed.



DEMONSTRATION.

The angle ACB, at the periphery, ſtanding upon the arch AQB, is equal to EOB, half the angle at the center, ſtanding upon the ſame arch; but EBO is equal to the difference of the given angle and a right one *(by conſtruction)* therefore ACB (EOB) is equal to the angle given. Q. E. D.

Method of calculation.

Draw CFG parallel to AB; then it will be, as the baſe AB : to the difference of ſegments CG (: : EB : CF) : : the ſine of the given angle at the vertex (EOB) :

to the ſine of (COF = CBG) the difference of the angles at the baſe; whence the angles themſelves are given.

After the ſame manner a ſegment of a circle may be deſcribed to contain a given angle, when that angle is greater than a right one, if, inſtead of BO being drawn above AB, it be taken on the contrary ſide.

PROBLEM V.

Having given the baſe, the perpendicular, and the angle at the vertex of any plane triangle, to conſtruct the triangle.

CONSTRUCTION.

Upon AB the given baſe (*ſee the preceding figure*) let the ſegment ACGB of a circle be deſcribed to contain the given angle, as in the laſt problem; take EF equal to the given perpendicular, and draw FC parallel to AB, cutting the periphery of the circle in C; join A, C and B, C, and the thing is done: the demonſtration whereof is evident from the laſt problem.

Method of calculation.

In the triangle EBO are given all the angles and the ſide EB, whence EO will be known, and conſequently OF (= DC — EO); then it will be as EB : OF : : the ſine of EOB (the given angle at the vertex) to the ſine of OCF, the complement of (COF or CBG) the difference of the angles at the baſe; whence theſe angles themſelves are likewiſe given.—This calculation is adapted to the logarithmic canon; but, by means of a table of natural ſines, the ſame reſult may be brought out by one proportion, only: for BE being the ſine of BOE, and OE and OF co-ſines of BOE and COF (anſwering to the equal radii OB and OC) it will therefore be, BE : EF : : *ſine* BOE (ACB) : *co-ſine* BOE + *co-ſine* COF; from which, by ſubtracting the co-ſine of BOE, the co-ſine of COF (= CBG) is found.

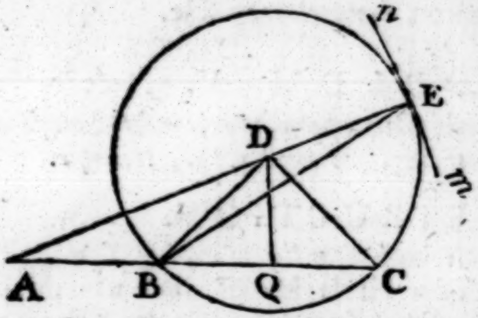
PROBLEM VI.

The angle at the vertex, the ſum of the two including ſides, and the difference of the ſegments of the baſe being given, to deſcribe the triangle.

CON-

CONSTRUCTION.

Draw the right line AC at pleaſure, in which take AB equal to the difference of the ſegments of the baſe, and make the angle CBE equal to half the ſupplement of the given angle; and from A to BE apply AE equal to the given ſum of the ſides; make the angle EBD = BED and let BD meet AE in D, and from the center D, with the radius DB, deſcribe the circle DBC, cutting AC in C, and join D, C; then will ACD be the triangle required.



DEMONSTRATION.

The angle EBD being = BED, therefore is DE = DB = DC, and conſequently AD + DC = AE. Moreover the angle CDE, at the center, is double to the angle CBE, at the periphery, both ſtanding upon the ſame arch CE; which laſt *(by conſtruction)* is equal to half the ſupplement of the given angle, therefore CDE is equal to the whole ſupplement, and conſequently ADC equal to the given angle itſelf. Q. E. D.

Method of calculation.

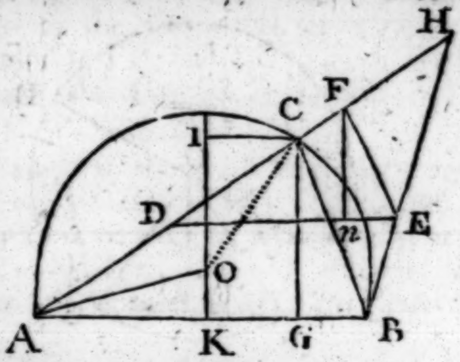
In the triangle ABE, are given the two ſides AB, AE, and the angle ABE, whence the angle A will be given; then in the triangle ABD will be given all the angles and the ſide AB, whence AD and DC (DB) will be alſo given.

PROBLEM VII.

The angle at the vertex, the ſum of the including ſides, and the ratio of the ſegments of the baſe being given; to determine the triangle.

CONSTRUCTION.

Let AG be to GB, in the given ratio of the ſegments of the baſe, and, upon the right-line AB, let a ſegment of a circle be deſcribed, capable of containing the given angle; draw GC perpendicular to AB, meeting the periphery in C; join A, C and C, B, and in AC, produced, take $CH = CB$; join B, H, and in HA, take HD equal to the given ſum of the ſides, draw DE parallel to AB, and EF to BC; then will DEF be the triangle required.



DEMONSTRATION.

Let Fn be perpendicular to DE. Whereas *(by conſtruction)* CH is equal to CB, and FE parallel to CB, therefore is $FE = FH$ *(Euc.* 4. 6.) and conſequently $FE + FD = HD$: alſo, becauſe FE is parallel to CB, therefore is the angle $DFE = ACB$: moreover, the triangles ABC, DEF, being equiangular, it will be, as $AG : GB :: Dn : nE$. Q. E. D.

Method of calculation.

From the center O, conceive AO and OC to be drawn; ſuppoſing KOI perpendicular, and CI parallel to AB: then it will be, as AK is to CI (KG) ſo is the ſine of AOK (= ACB, *ſee Prob.* 4.) to the ſine of COI, the difference of the angles ABC and BAC; which are both given from hence, becauſe their ſum is given by the queſtion: therefore in the triangle DHE are given all the angles and the ſide HD, whence the baſe DE will be known.

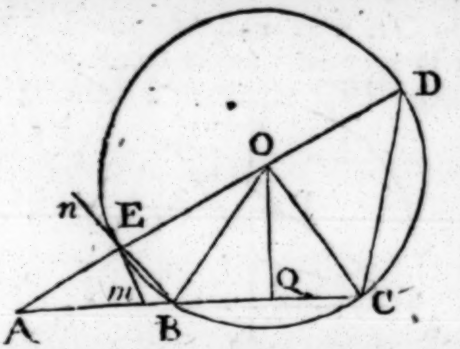
PROBLEM VIII.

Having the angle at the vertex, the difference of the including ſides, and the difference of the ſegments of the baſe, to deſcribe the triangle.

CONSTRUCTION.

Take AB equal to the difference of the ſegments of the baſe, and make the angle ABn equal to half the

given angle; from A to Bn apply AE = the difference of the ſides; produce AE, and make the angle EBO = BEO, and let BO meet AE, produced in O, and from the center O, at the diſtance of OB, deſcribe the circumference of a circle, cutting AB produced in C, join O, C; then is AOC the triangle ſought.



DEMONSTRATION.

Becauſe the angle EBO is = BEO *(by conſtruction)*; therefore is EO = BO = CO, and conſequently AO — OC = AE. Furthermore becauſe the angle AOC is double to ADC, and ADC = ABE *(Euc. Corol.* 22. 3.) therefore is AOC alſo double to ABE. Q. E. D.

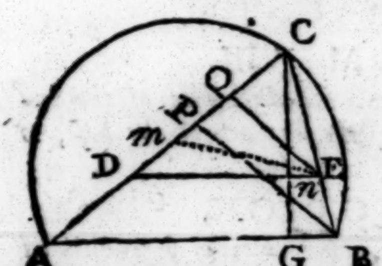
Method of calculation.

The two ſides AB AE, and the angle ABE being given, the angle A will from thence be found; then in the triangle ABO will be given all the angles and the ſide AB, whence OB (OC) and OA will be known.

PROBLEM IX.

The angle at the vertex, the difference of the including ſides, and the ratio of the ſegments of the baſe being given, to determine the triangle.

CONSTRUCTION.



Let AG be to GB, in the given ratio of the ſegments of the baſe, and upon the right-line AB let a ſegment of a circle ACB be deſcribed *(by Prob.* 4.) capable of the given angle; draw GC perpendicular to AB, meeting the periphery in C, and join A, C and B, C; in AC take AP = BC, and draw BP; alſo, in AC, take CQ equal to the given difference of the ſides, drawing QE parallel to PB, and ED to BA; then will CDE be the triangle which was to be deſcribed.

DEMONSTRATION.

The angle DCE is equal to the given angle by conſtruction; alſo EQ being parallel to BP, DE to AB, and AP = BC, therefore muſt DQ = EC (*Euc.* 4. 6.) and conſequently DC — EC = CQ. Moreover, if CG be ſuppoſed to cut DE in *n*, then D*n* : E*n* : : AG : GB. Q. E. D.

Method of calculation.

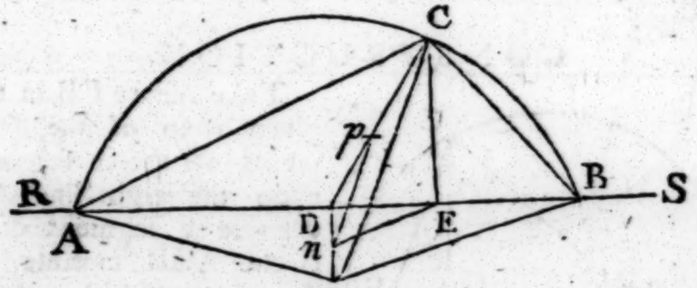
Let C*m* be equal to CE, and let E*m* be drawn. It will be, as AB is to AG — BG, ſo is the ſine of ACB to the ſine of the difference of CBA and CAB (*by Prob.* 4.); then in the triangle DE*m* will be given all the angles and the ſide D*m*, whence DE will be given.

PROBLEM X.

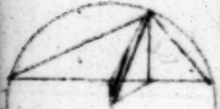
The angle at the vertex, the perpendicular, and the difference of the ſegments of the baſe being given, to conſtruct the triangle.

CONSTRUCTION.

Draw RS at pleaſure, in which take DE equal to half the difference of the ſegments of the baſe, and make EC perpendicular to RS and equal to the given perpendicular, and the angle DE*n* equal to the difference between the given angle and a right one; join D, C, and draw D*n*O parallel to CE, and in DC take the



point *p*, ſo that *np* (when drawn) may be equal to *n*E; draw CO parallel to *np*, meeting D*n*O in O; and upon O as a center, with the radius OC, deſcribe the circle BCA, cutting RS in B and A; join A, C and B, C, and the thing is done.



DEMONSTRATION.

Join O, B and O, A: ſince OC is parallel to *pn*, therefore is OC : DO : : *pn* : *n*D, or OB : DO : : *n*E : *n*D; and conſequently the triangle OBD ſimilar to the triangle *n*ED (*by Euc.* 7. 6.) Therefore, ſeeing the angle DE*n* is (*by conſtruction*) equal to the exceſs of the given angle above a right one, ACB muſt be equal to the angle given (*by Prob.* 4.) Moreover, ſince AD is = DB, AE — BE will be equal to 2DE, which is the given difference of the ſegments (*by conſtruction*). Q. E. D.

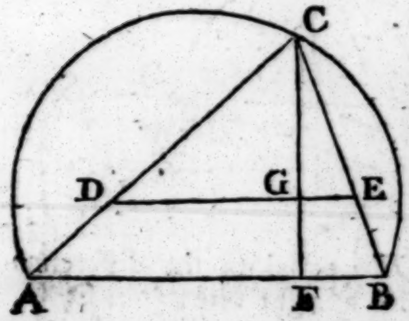
Method of calculation.

In the triangle CDE, right-angled at E, are given both the legs DE and EC, whence the angle EDC will be known, and conſequently ODC; then, as the radius is to the ſine of DBO (: : OB : DO : : OC : DO) ſo is the ſine of ODC to the ſine of OCD; whence DOC, the difference of the angles ABC, BAC, (*ſee Prob.* 4.) is alſo given, and from thence the angles themſelves.

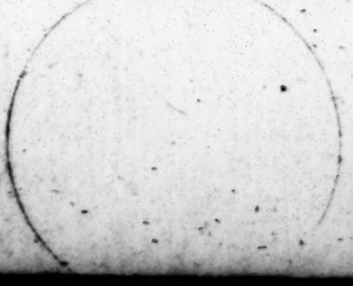
PROBLEM XI.

The angle at the vertex, the perpendicular, and the ratio of the ſegments of the baſe being given, to conſtruct the triangle.

CONSTRUCTION.



Take AF to FB in the given ratio of the ſegments of the baſe, and upon the right-line AB deſcribe a ſegment of a circle ACB capable of the given angle; make FC perpendicular to AB meeting the circumference of the circle in C, in which take



take CG equal to the given perpendicular; draw DGE parallel to AB, meeting AC and CB in D and E; and then DCE will be the triangle required.

DEMONSTRATION.

Becauſe of the parallel lines DE and AB, it will be as AF : DG (: : CF : CG) : : FB : GE, or AF : FP : : DG : GE; whence it appears, that DG and GE are in the ratio given. Alſo the angle DCE and the perpendicular CG are reſpectively equal to the given angle and perpendicular, by conſtruction. Q. E. D.

Method of calculation.

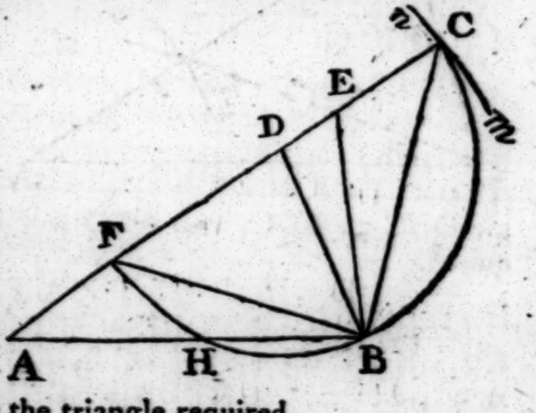
As AB is to AF — BF (*ſee Prob.* 4.) ſo is the ſine of ACB to the ſine of the difference of A and B; whence both A and B will be given, becauſe their ſum, or the angle at the vertex, is given: then in the triangles DGC, EGC, will be given all the angles and the perpendicular CG, whence the ſides will alſo be known.

PROBLEM XII.

The baſe, the ſum of the ſides, and the difference of the angles at the baſe being given, to deſcribe the triangle.

CONSTRUCTION.

At the extremity of the baſe AB, erect the perpendicular BE, and make the angle EBC equal to half the given difference of the angles at the baſe; from the point A, to BC, apply AC equal to the ſum of the ſides; and make the angle CBD = BCA; then will ABD be the triangle required.



DEMONSTRATION.

From the center D, with the radius CD, deſcribe the ſemi-circle CHF, and join F, B. Then, whereas by conſtruction the angle CBD is = BCD, therefore is DB = DC; whence it appears that AD + DB is = AC, and that the ſemi-circle muſt paſs through the point B: therefore, the angle CBF, ſtanding in a ſemi-circle, being a right angle, and therefore = ABE, let FBE, which is common, be taken away, and there will remain ABF ± EBC; but DF being equal to DB, it is manifeſt that ABF (EBC) is equal to half the difference of the angles ABD and DAB. Q. E. D.

Method of calculation.

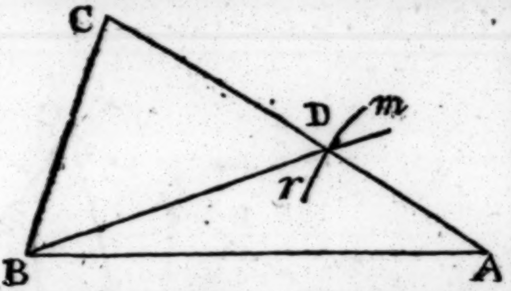
As the ſum of the ſides (AC) is to the baſe (AB) ſo is the ſine of ABC, or of the complement of half the given difference, to the ſine of (C) half the angle at the vertex; whence the other angles BAD and ABD are alſo given.

PROBLEM XIII.

The baſe, the difference of the ſides, and the difference of the angles at the baſe, being given, to determine the triangle.

CONSTRUCTION.

At the extremity B of the given baſe AB, make the angle ABD equal to half the given difference of the angles at the baſe; and from A to BD apply AD = the difference of the ſides; draw ADC, and make the angle DBC = BDC, and ABC will be the triangle required.



DEMONSTRATION.

Becauſe the angle DBC is = BDC, CD will be = CB, and AC will exceed BC by AD. Moreover, ſince A + ABD = (CDB) CBD (*Euc.* 32. 1.) therefore is A + 2ABD (= CBD + ABD) = ABC, and conſequently ABC — A = 2ABD, equal to the difference given. Q. E. D.

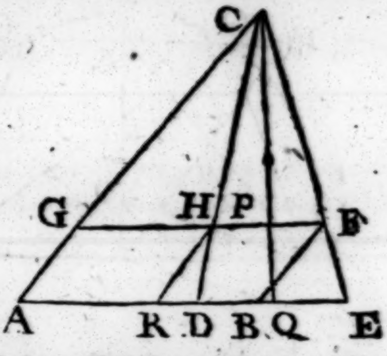
Method of calculation.

In the triangle ABD are given the two ſides, AB and AD, and the angle ABD, whence the angles A and ADB will be given, and from thence the angles CBA and ACB.

PROBLEM XIV.

The difference of the angles at the baſe, the ratio of the ſides, and either the baſe, the perpendicular, or the difference of the ſegments of the baſe being given, to deſcribe the triangle.

CONSTRUCTION.



Draw AC at pleaſure, and make the angle ACD equal to the given difference of the angles at the baſe, and take CD to CA in the given ratio of the ſides; draw ADE, upon which let fall the perpendicular CQ, take QE equal to QD, and join E, C; then, if the baſe be given, let AB be taken equal thereto, and draw BF parallel to CA (meeting CE in F) and FG parallel to EA; but if the perpendicular be given, let CP be taken equal thereto, and through P draw FPG parallel to AE; laſtly, if the difference of the ſegments of the baſe be given, then let AR be taken equal to that difference, draw RH parallel to CA, and FHG to EA; then will CFG be the triangle required.

DEMONSTRATION.

Since $QE = QD$, and the angle $EQC = DQC$, therefore is $CE = CD$, and the angle $E = QDC = A + ACD$ (*Euc.* 32. 1.) and therefore $E - A = ACD$; whence, by reaſon of the parallel lines AE, GF, &c. we have $GFC - FGC = ACD$, alſo $FG = AB$, $GH = AR$, and $CF : CG :: CE\ (CD) : CA$. Q. E. D.

Method of calculation.

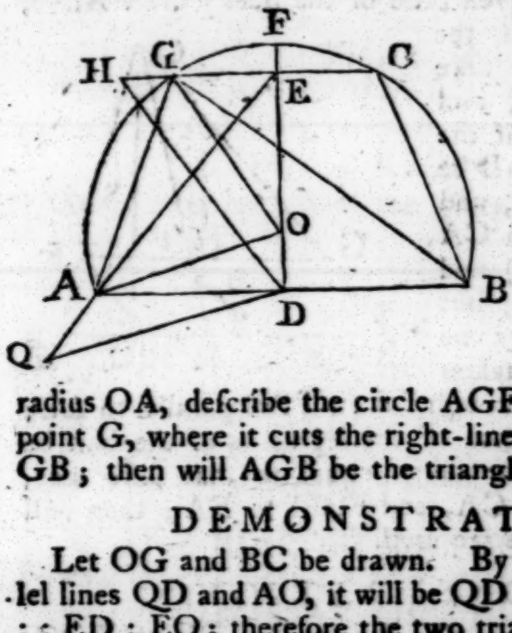
Let CA and CD be expreſſed by the numbers exhibiting the given ratio of the ſides: then in the triangle ACD will be given two ſides and the included angle ACD; whence the angle CAE (CGF) and CEA (CFG) will be given, and from thence the ſides CG and CF.

PROBLEM XV.

The baſe, the perpendicular, and the difference of the angles at the baſe being given, to conſtruct the triangle.

CONSTRUCTION.

Biſect the given baſe AB by the perpendicular DF, in which take DE equal to the given height of the triangle; draw CEGH parallel to AB, and make the angle EDH equal to the given difference of the angles at the baſe; draw EAQ, and take Q therein, ſo that QD = DH; and, parallel to QD, draw AO, meeting DE in O; upon O, as a center, with the radius OA, deſcribe the circle AGFCB, and from the point G, where it cuts the right-line CH, draw GA and GB; then will AGB be the triangle required.



DEMONSTRATION.

Let OG and BC be drawn. By reaſon of the parallel lines QD and AO, it will be QD (DH) : AO (OG) : : ED : EO; therefore the two triangles EHD, EGO, having one angle, E, common, and the ſides about the other angles D and O proportional, are equiangular (*Euc.* 7. 6.) and conſequently EOG = EDH. Moreover, becauſe DOEF is perpendicular both to AB and GC, and AD equal to BD, it is evident that the circle paſſes through the point B, and that the arches FC, FG, as well as the angles ABC, BAG, are equal; and conſequently that the angle GBC is the difference of the angles BAG, ABG: but this difference GBC is equal to EOG, or EDH (*Euc.* 20. 3.) that is, equal to the difference given. Q. E. D.

Method of calculation.

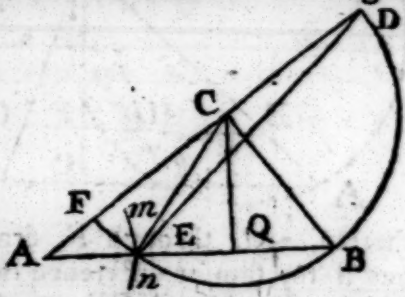
Firſt, in the right-angled triangle AED are given both the legs AD and DE, whence the angle DEA will be given; then it will be, as the radius is to the ſine of the angle H, the complement of the given difference (: : DH : DE : : DQ : DE) ſo is the ſine of DEA to the ſine of Q; whence AOE (QDE) will alſo be given; from which take GOE, and there will remain AOG, equal to twice ABG, the leſſer angle at the baſe.

PROBLEM XVI.

The ſum of the ſides, the difference of the ſegments of the baſe, and the difference of the angles at the baſe, being given, to deſcribe the triangle.

CONSTRUCTION.

Make AD equal to the ſum of the ſides, and the angle ADE equal to half the difference of the angles at the baſe; from A to DE apply AE equal to the given difference of the ſegments of the baſe; make the angle CED = EDC, and from the point C, where EC cuts AD, with the radius EC, deſcribe the ſemi-circle FEB, cutting AE, produced in B; join B, C, and the thing is done.



DEMONSTRATION.

Upon AB let fall the perpendicular CQ.

Becauſe EQ is = BQ (*Euc.* 3. 3.) therefore will AQ — BQ = AE: alſo, becauſe the angles CED, EDC, are equal (*by conſtruction*) CD will be = CE = CB, and conſequently AC + CB = AD. Moreover, ABC

— BAC = BEC — BAC = ACE (*Euc.* 32. 1.) = 2ADE (*Euc.* 20. 3.) Q. E. D.

Method of calculation.

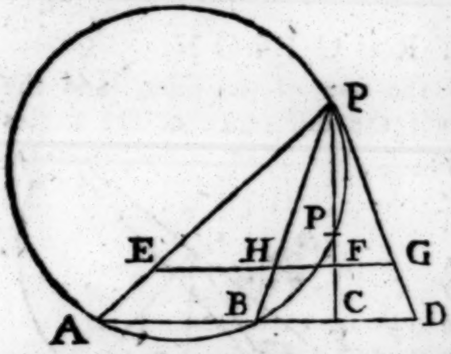
In the triangle ADE are given the ſides AD, AE, and the angle D, whence the angle A will be given; then in the triangle ACE are given all the angles and the ſide AE, whence AC and CB (CE) will be given likewiſe.

PROBLEM XVII.

The difference of the angles at the baſe, the ratio of the ſegments of the baſe, and either the ſum of the ſides, the difference of the ſides, or the perpendicular being given, to conſtruct the triangle.

CONSTRUCTION.

Let AC be to BC in the given ratio of the ſegments of the baſe; and upon AB let a ſegment of a circle BPA be deſcribed (*by Problem* 4) to contain an angle equal to the difference of the angles at the baſe; raiſe CP perpendicular to AC, cutting the periphery of the circle in P, and in AC produced, take CD = CB, and draw PA, PB and PD: then, if the perpendicular be given, take PF equal thereto, and, through F, draw EFG parallel to AD; but if the ſum or difference of the ſides be given, let a fourth proportional PE, to AP ± PD, AP and the ſaid ſum or difference be taken, and draw EFG as above; then will PEG be the triangle required.



DEMONSTRATION.

Since CP is perpendicular to AD, and CD = CB, the angle D will be equal to DBP = A + BPA: whence, becauſe EG is parallel to AD, PGE will be = PEG + BPA (*Euc.* 29. 1.) and conſequently PGE + PEG =

= ABP, which, by conſtruction, is equal to the given difference of the angles at the baſe.

Again, by reaſon of the parallel lines AD and EG, it will be, EF : FG : : AC : (BC) CD. Likewiſe, for the ſame reaſon, AP ± PD : PA : : PE ± PG : PE : : given ſum or diff. of ſides : PE *(by conſtruction)* and conſequently PE ± PG = the ſaid given ſum or difference. *Q. E. D.*

Method of Calculation.

Firſt, it will be as AB is to AD, ſo is the ſine of APB to the ſine of APD *(by Prob. 4.)*; then in the triangle PGE will be given all the angles, and either the perpendicular, or the ſum or difference of the ſides, whence the ſides themſelves are readily determined.

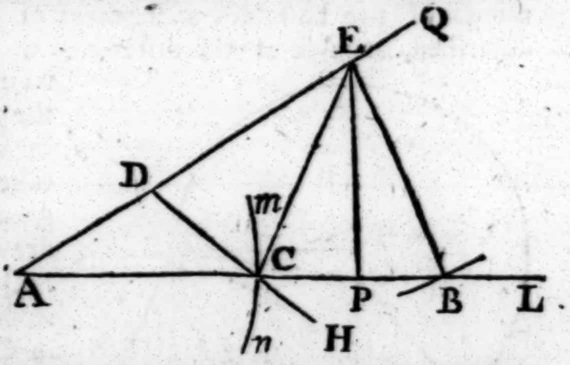
Note. The perpendicular cutting the circle in two points, indicates that this problem is capable of two different ſolutions.

PROBLEM XVIII.

The difference of the ſides, the difference of the ſegments of the baſe, and the difference of the angles at the baſe being given, to deſcribe the triangle.

CONSTRUCTION.

Draw the indefinite line AQ, in which take AD equal to the given difference of the ſides, and make the angle QDH equal to the complement of half the difference of the angles at the baſe; from A to DH apply AC = the given difference of the ſegments; and, having produced the ſame to L, make the angle DCE equal to CDE, and let CE meet AQ in E, and upon the center E, with the radius EC, deſcribe an arch, cutting AL in B; join EB, ſo ſhall AEB be the triangle required.



DEMON-

DEMONSTRATION.

Upon AB let fall the perpendicular EP.

Becauſe the angle DCE = CDE, therefore is ED = EC, and conſequently AE — EB (= AE — EC = AE — ED) = AD. Alſo, ſince EB = EC, therefore will PB = PC, and conſequently AP — BP (AP — PC) = AC. Moreover, the angle EBC being = ECB (*Euc.* 5. 1.) and ECB — A = CEA (*Euc.* 32. 1.) it is plain that EBC — A = CEA equal to the given difference, becauſe the triangle EDC is iſoſceles, and the angle at the baſe equal to the complement of half the ſaid difference, by conſtruction. Q. E. D.

Method of calculation.

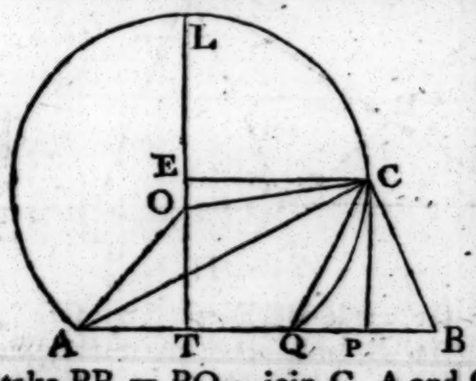
In the triangle ADC are given two ſides and the angle ADC, whence the angle A will be known; then in the triangle ACE will be given all the angles and the ſide AC, whence AE and CE (BE) will alſo become known.

PROBLEM XIX.

The perpendicular, the difference of the angles at the baſe, and the difference of the ſegments of the baſe being given, to conſtruct the triangle.

CONSTRUCTION.

Upon AQ, equal to the given difference of the ſegments of the baſe, let a ſegment of a circle QCA be deſcribed, capable of the difference of the angles at the baſe; biſect AQ with the perpendicular TL, in which let TE be taken equal to the given perpendicular; draw EC parallel to AQ, cutting the periphery of the circle in C; alſo draw CP perpendicular to AQ, and in AQ produced take PB = PQ; join C, A and C, B; then will ACB be the triangle required.



DEMON-

DEMONSTRATION.

Since *(by conſtruction)* CP is perpendicular to QB, and PB equal to PQ, thence will the angle B = PQC, and B (PQC) — BAC = ACQ = difference of angles given: alſo, for the ſame reaſon, will CP = TE, and AP — BP = AP — PQ = AQ. Q. E. D.

Method of calculation.

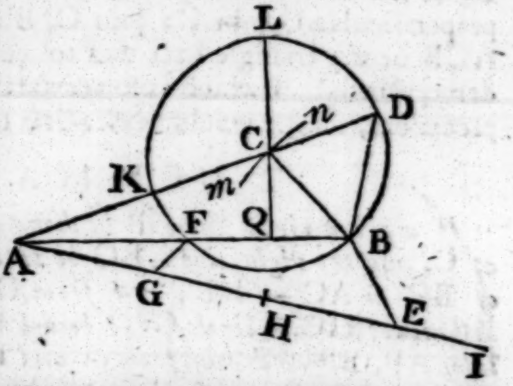
From the center O, conceive OA and OC to be drawn: then in the triangle AOT will be given all the angles and the ſide AT, whence OT and OE will be given; then it will be as AT : OE : : ſine of AOT (ACQ): ſine of OCE; whence all the angles in the figure are given.

PROBLEM XX.

The ſegments of the baſe, and the ſum of the ſides of any plane triangle being given, to determine the triangle.

CONSTRUCTION.

From the greater ſegment AQ, take QF equal to the leſſer ſegment BQ; make QL perpendicular to AB, and draw AI, making any angle with AB at pleaſure, in which take AE equal to the given ſum of the ſides, and join B, E; make the angle AFG = AEB, and biſect EG in H, and from B as a center, with the radius EH, deſcribe *m*C*n*, cutting the perpendicular QL in C; join C, A and C, B, and the thing is done.



DEMONSTRATION.

From the center C, with the radius CB, let the circle BDLKF be deſcribed; and let AC be produced to meet its periphery in D. By reaſon of the ſimilar triangles AEB, AFG, it will be as AE : AB : : AF : AG, whence AG × AE = AF × AB; but *(by Euc.* 37. 3.)

AF × AB = AK × AD; therefore is AG × AE = AK × AD: whence, as EG and DK are equal, by conſtruction, it is evident that AG and AK, as well as AE and AD, muſt be equal. *Q. E. D.*

Method of calculation.

As AE : AB : : AF : AG; which taken from AE, and the remainder divided by 2, gives BC (EH) the leſſer ſide of the triangle.

PROBLEM XXI.

The ſegments of the baſe and the difference of the ſides being given, to deſcribe the triangle.

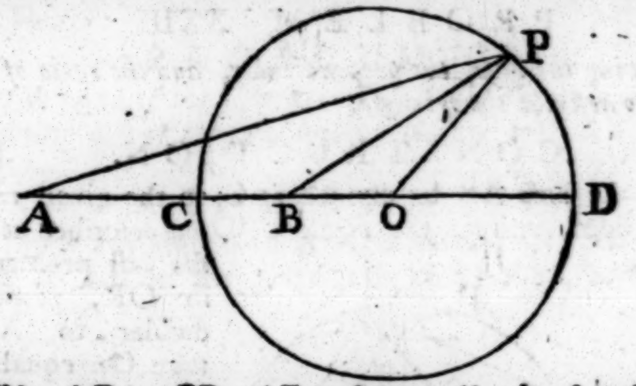
CONSTRUCTION.

Take AF equal to the difference of the given ſegments AQ, BQ, *(ſee the preceding figure)* and draw AI making any angle with AB at pleaſure, in which take AG equal to the given difference of the ſides; join F, G, and make the angle ABE = AGF, and from the center B, at the diſtance of $\frac{1}{2}$EG, deſcribe *n*C*m*, cutting the perpendicular QL in C; join C, B and C, A, then will ACB be the triangle that was to be conſtructed. The demonſtration of which is ſo very little different from the precedent, that it would be needleſs to give it here.

LEMMA.

If a given right-line AB *be divided in any given ratio, at* C, *and the right-line* CBO *be taken to* AC *in the ratio of* BC *to* AC — BC; *and from* O *as a center, at the diſtance of* OC, *a circle* CPD *be deſcribed, and two right-lines* AP, BP *be drawn from* A *and* B, *to meet any where in the periphery thereof; I ſay theſe lines will be to one another (every where) in the given ratio of* AC *to* CB.

For, ſince CO : AC : : BC : AC — BC, therefore by compoſition, CO : AO : : BC : AC, and by permutation, CO : BC : : AO : AC; whence, by diviſion, CO : BO : : AO : CO, or PO : BO : : AO : PO: wherefore, ſeeing the ſides of the triangles POB, AOP, about the common angle O, are proportional, thoſe triangles muſt be ſimilar (*Euc.* 6. 6.) and therefore the other ſides alſo proportional, that is, PO



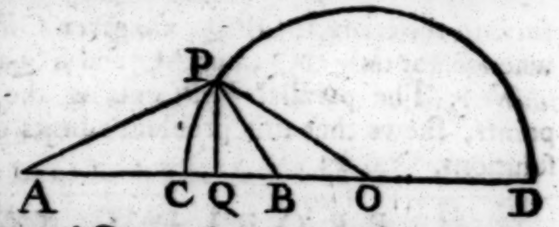
(CO) : AO : : BP : AP; whence (*by the ſecond ſtep*) BC : AC : : BP : AP. Q. E. D.

PROBLEM XXII.

The ſegments of the baſe, and the ratio of the ſides being given, to determine the triangle.

CONSTRUCTION.

Let AQ, and QB be the ſegments of the baſe; and let the whole baſe AB be divided at C, in the given ratio of the ſides; take CO to AC, as BC to AC — BC, and with the radius CO deſcribe the circle CPD, and raiſe QP perpendicular to AO, meeting the periphery in P; join A, P and B, P; then will ABP be the triangle required. The demonſtration of which is manifeſt from the preceding lemma.



Method of calculation.

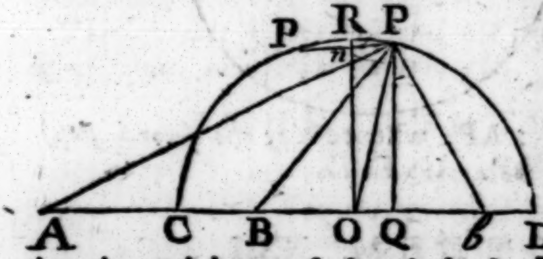
Since the ratio of AC to CB, and the length of the whole line AB are given, thence will AC and CB be given, and conſequently OC $\left(\frac{AC \times BC}{AC - BC}\right)$ from whence the perpendicular PQ $(= \sqrt{CQ \times DQ})$ is likewiſe given.

PROBLEM XXIII.

Having the baſe, the perpendicular, and the ratio of the ſides, to deſcribe the triangle.

CONSTRUCTION.

Let the baſe AB be divided at C, in the given ratio of the ſides, and let the circle CPD be deſcribed as in the laſt problem; in OR, perpendicular to AD, take O*n* equal to the given perpendicular, and, thro' *n*, draw P*n*P parallel to AD, cutting the periphery of the circle in P; join P, A and P, B, and the thing is done. The truth of this is alſo evident from the preceding lemma.



Method of calculation.

Upon AD let fall the perpendicular PQ, and join O, P: then PO $\left(= \frac{AC \times BC}{AC - BC}\right)$ will be given; therefore, in the triangle OPQ, are given OP and PQ, from whence not only OQ, but AQ and BQ are alſo given.

Note. The parallel P*n*P cutting the circle in two points, ſhews that this problem admits of two different ſolutions.

PROBLEM XXIV.

The difference of the ſegments of the baſe, the perpendicular and the ratio of the ſides being given, to conſtruct the triangle.

CONSTRUCTION.

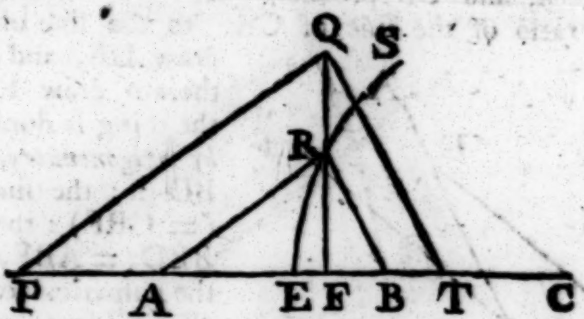
Let AB be the difference of the ſegments of the baſe (*ſee the laſt figure*) and let every thing be done as in the preceding problem: take Q*b* = QB, and join P, *b*; then will A*b*P be the triangle required. The reaſons of which are obvious from what has been ſaid already; and the numerical ſolution is alſo evident from the laſt problem.

PROBLEM XXV.

The ratio of the ſegments of the baſe, the perpendicular, and the ratio of the ſides being given, to conſtruct the triangle.

CONSTRUCTION.

Draw any right-line ABC at pleaſure, in which take AE to EB in the given ratio of the ſides, and AF to FB in the given ratio of the ſegments of the baſe, and make FQ perpendicular to AB, and equal to the given height of the triangle; make alſo EC : AE : : BE : AE — BE, and with the radius CE deſcribe the circle ERS, and from the point R where it interſects the perpendicular FQ draw RA and RB, and draw QP and QT parallel to RA and RB; then will PQT be the triangle that was to be deſcribed.



DEMONSTRATION.

By the foregoing lemma, AR : BR : : AE : BE; therefore, by reaſon of the parallel lines, it will be QP : QT (: : RA : RB) : : AE : BE. And, for the ſame reaſon, PF : TF : : AF : BF. Q. E. D.

Method of calculation.

Having aſſumed AB at pleaſure, there will be given BE, AE, BF and CE $\left(= \frac{AE \times BE}{AE - BE}\right)$ whence RF $\left(= \sqrt{EF \times \overline{CE + CF}}\right)$ is alſo given; then, in the right-angled triangle BRF, will be given both the legs

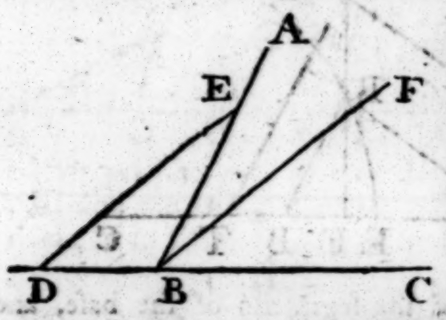
BF and RF, whence the angle B is given; laſtly, in the triangle FQT will be given all the angles and the ſide FQ, whence QT and TF will be given, and conſequently PQ and FP.

PROBLEM XXVI.

To divide a given angle ABC *into two parts* CBF, ABF, *ſo that their ſines may obtain a given ratio.*

CONSTRUCTION.

In BA, and CB produced, take BE and BD in the given ratio of the ſine of CBF to the ſine of ABF; draw DE, and parallel thereto draw BF, and the thing is done. For, *by trigonometry*, BE : BD : : the ſine of D (= CBF) : the ſine of BED (= ABF). Hence the numerical ſolution is alſo evident; ſince it will be, as the ſum of BE and BD is to their difference, ſo is the tangent of half the given angle ACB to the tangent of half the difference of the two required parts FBC and FBA.

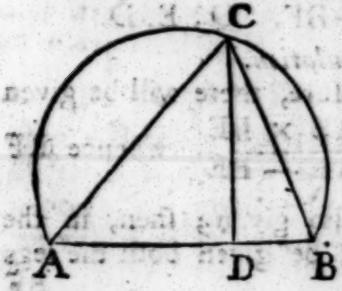


PROBLEM XXVII.

To divide an angle given into two parts, ſo that their tangents may be to each other in a given ratio.

CONSTRUCTION.

Take any two right lines AD, BD, which are in the ratio given, and upon the whole compounded line AB let a ſegment of a circle BCA be deſcribed, capable of the angle given; make DC perpendicular to AB, meeting the periphery in C, and draw AC and BC, then will ACD and BCD be the two angles required.



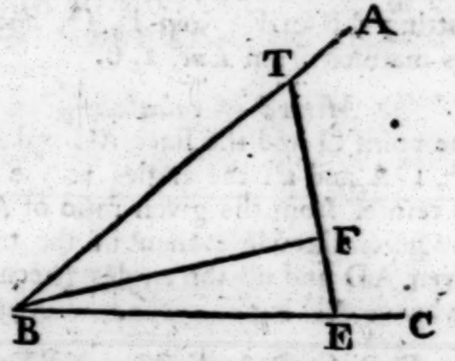
The reaſon of which is evident, at one view, from the conſtruction. The method of ſolution is alſo very eaſy; for it will be, as AB is to AD—DB, ſo is the ſine of ACB to the ſine of B—A (*ſee Problem* 4.) whence B and A, and alſo BCD and ACD are given.

PROBLEM XXVIII.

To divide a given angle ABC *into two parts, ſo that their ſecants may obtain a given ratio.*

CONSTRUCTION.

Take BE to BT in the given ratio of the ſecants; join T, E, and let BF be drawn perpendicular to ET, and the thing is done. The truth of which is manifeſt, from the conſtruction.



Method of calculation.

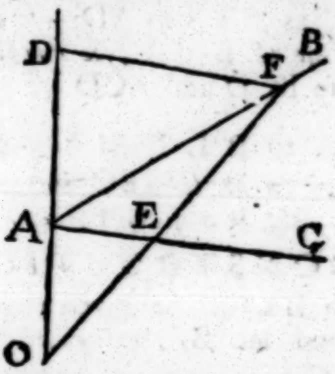
The angle EBT and the ratio of the ſides BE, and BT being given, the angles E and T will alſo be given, and conſequently their complements EBF and FBT.

PROBLEM XXIX.

From a given point O, *to draw a right-line* OF, *to cut two right-lines* AC, AB, *given by poſition, ſo that the parts thereof,* OE, OF, *intercepted between that point and thoſe lines, may be to one another in a given ratio.*

CONSTRUCTION.

From O, through A, the point of concourſe of BA and CA, let OAD be drawn, in which take AD to AO in the given ratio of FE to EO, and draw DF parallel



to AC, cutting AB in F; join F, O, and the thing is done; as is manifeſt from *Euc.* 2. 6.

Method of calculation.

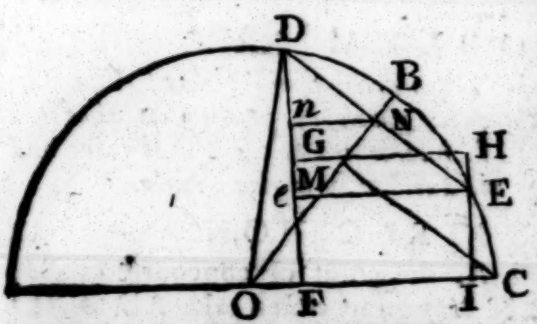
Since the point O and the lines AC and AB are given by poſition, OA and all the angles at the point A are given; therefore, from the given ratio of AD and AO, AD will be given likewiſe; then in the triangle DAF will be given AD and all the angles (becauſe FDA = CAO); whence AF is alſo given.

PROBLEM XXX.

To divide a given arch CD *into two ſuch parts, that the rectangle under their ſines may be of a given magnitude.*

CONSTRUCTION.

Upon the radius OC let fall the perpendicular DF, in which (produced if need be) take FG = $\frac{1}{2}$OC, and thereon conſtitute a rectangle FIHG equal to the given rectangle; and ſuppoſing HI to cut the circumference in E, draw OB to biſect DE; then will CB and DB be the parts required.



DEMON-

DEMONSTRATION.

Draw CM, and DNE perpendicular to the radius OB, and Nn and Ee perpendicular to DF.

It is evident, by conſtruction, that the triangles OCM, and DNn are ſimilar (becauſe Nn is parallel to CO, and ND to CM) therefore $OC : CM :: DN : Nn$ $(= \frac{1}{2}Ee)$, and conſequently $CM \times DN = OC \times \frac{1}{2}Ee = \frac{1}{2}OC \times Ee = FG \times Ee =$ the given rectangle by conſtruction. *Q. E. D.*

Method of calculation.

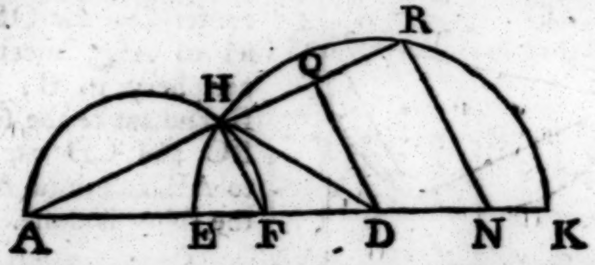
Dividing the meaſure of the given rectangle by half the radius, FI will be given, which added to OF, the co-ſine of CD, gives the co-ſine (OI) of CE, the difference of the two parts; whence the parts themſelves will be known.

PROBLEM XXXI.

Having the ratio of the ſines, and the ratio of the tangents of two angles, to determine the angles.

CONSTRUCTION.

Let AD be to ED in the given ratio of the ſines, and AD to FD in the given ratio of the tangents; and about the center D, with the interval DE, let the ſemi-circle ERK be deſcribed; and, upon AF, deſcribe another ſemi-circle, cutting the former in H, and through H draw AR, and join H, D; then will DHR and DAR be the two angles required.



DEMONSTRATION.

Join F, H, and draw DQ perpendicular to AR.

The angle AHF, ſtanding in a ſemi-circle, being a right one, the lines FH and DQ are parallel (*by Euc.* 27. 1.)

and therefore AD : FD : : AQ : HQ : : tang. of DHQ : tang. DAQ. Likewiſe DA : DE (DH) : : ſine of DHQ : ſine of DAQ, as was to be ſhewn.

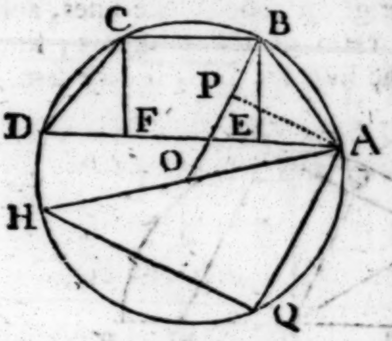
Method of calculation.

If AR be ſuppoſed to meet the periphery in R, and RN be drawn parallel to HF, meeting AK in N; then will DN = DF, and AN : AR : : AF : AH; but (*by Euc.* 37. 3.) AR : AK : : AE : AH; whence, by compounding the terms of theſe two proportions, *&c.* AN : AK : : AF × AE : AH^2; whence AH, as well as AD and DH, being known, the angles A and K will alſo be known.

PROBLEM XXXII.

To draw from a point A *in the circumference of a given circle, two ſubtenſes* AB *and* AD, *which ſhall be to one another in the given ratio of m to n, and cut off two arches* AB *and* ABD, *in the ratio of* 1 *to* 3.

CONSTRUCTION.



Draw the diameter AH, and take the ſubtenſe AQ, in proportion thereto, as *n* — *m* to 2*m*; from the center O draw OB parallel to AQ, meeting the periphery in B; join A, B, and make the ſubtenſes BC and CD each equal to AB, and draw AD, and the thing is done.

DEMONSTRATION.

Join H, Q, and draw BE and CF perpendicular to AD.

The angle AOB (QAH) at the center, ſtanding upon the arch AB, is equal to the angle BAD at the circumference, ſtanding upon double that arch; therefore, AQH being equal to AEB or a right angle (*Euc.* 31. 3.) the triangles AQH, AEB muſt be equiangular, and conſequently AB : AE : : AH : AQ; but, by conſtruction,

ſtruction, AH : AQ : : $2m : n-m$, whence AB : AE : : $2m : n-m$, or AB : 2AE : : $2m : 2n-2m$; therefore (*by compoſition*) AB : AB + 2AE (: : $2m : 2n$) : : $m : n$. But AB being = BC = CD, EF is = BC = AB, DF = AE, and AD = 2AE + AB. Hence AB : AD : : $m : n$. Q. E. D.

Method of calculation.

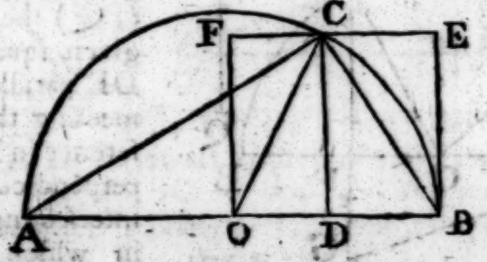
Let AP be perpendicular to OB; then, becauſe of the ſimilar triangles OAP, AHQ, it will be as AO : OP (: : AH : AQ) : : $2m : n-m$ (*by conſtruction*) therefore OP = $\frac{\overline{n-m}\times AO}{2m}$, BP = AO — OP) = $\frac{\overline{3m-n}\times AO}{2m}$, and conſequently AB ($\sqrt{2AO\times BP}$) = $\sqrt{\frac{3m-n}{m}}\times AO$; whence AD is alſo given.

PROBLEM XXXIII.

The area and hypothenuſe of any right-angled plane triangle being given, to deſcribe the triangle.

CONSTRUCTION.

Upon the given hypothenuſe AB, as a diameter, let the ſemi-circle ACB be deſcribed, and upon OB, equal to half AB (*by Euc.* 41. 1.) conſtitute the rectangle OE



equal to the given area of the triangle, and let the ſide thereof, EF, cut the periphery of the circle in C; join A, C, and B, C, and the thing is done.

DEMONSTRATION.

The triangle ABC, ſtanding upon the whole diameter AB, is equal to the rectangle OE, of the ſame altitude, ſtanding upon half AB (*by Euc.* 41. 1.) which laſt (*by conſtruction*) is equal to the area given.

Method of calculation.

Join O, C, and let CD be perpendicular to AB; then it will be, as AO^2 $(AO \times OC) : AO \times DC$ $(: : OC : DC) : :$ radius : ſine of DOC; which, in words, gives this theorem.

As the ſquare of half the hypothenuſe of any right-angled plane triangle is to the area, ſo is the radius to the ſine of double the leſſer of the two acute angles.

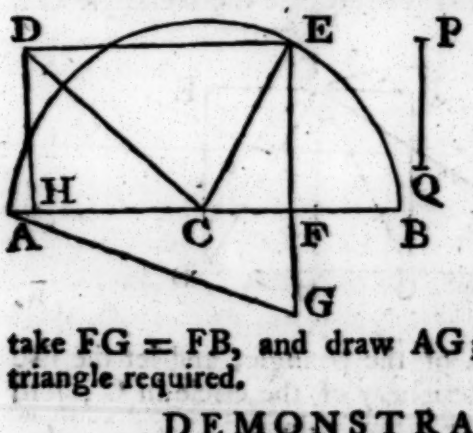
N. B. Since no ſine can be greater than the radius, it is plain, that, if the ſquare of half the hypothenuſe be not given greater than the area of the triangle, the problem will become impoſſible; in which caſe the ſide EF, inſtead of cutting, will paſs quite above the circle.

PROBLEM XXXIV.

To deſcribe a right-angled triangle, whoſe area ſhall be equal to a given ſquare, and the ſum of its two legs equal to a given right-line AB.

CONSTRUCTION.

Upon AB let a ſemi-circle be deſcribed; make ACD = half a right angle, and CD = twice (PQ) the ſide of the given ſquare; draw DE parallel to AB, meeting the circumference in E, and EF perpendicular to AB, interſecting AB in F, in which produced take FG = FB, and draw AG; ſo ſhall AFG be the triangle required.



DEMONSTRATION.

It is evident that AF + FG = AB; and alſo that the area AFG $= \frac{1}{2}AF \times FG = \frac{1}{2}AF \times FB = \frac{1}{2}FE^2$ ($= \frac{1}{2}DH^2$) $= \frac{1}{4}CD^2 = PQ^2$. Q. E. D.

Method of calculation.

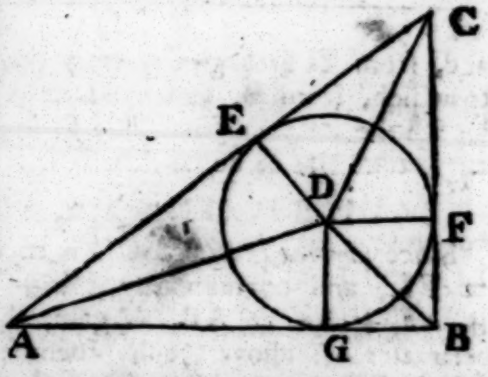
If the radius CE be drawn, in the right-angled triangle CEF, there will be given CE ($= \frac{1}{2}AB$) and $\overline{EF}\rceil^2$ ($= 2PQ^2$) whence CF ($= \sqrt{\frac{1}{4}AB^2 - 2PQ^2}$) will be known, and, from thence, both AF and FG.

LEMMA.

The area of any right-angled triangle, ABC, *is equal to a rectangle under half its perimeter and the exceſs of that half perimeter above the hypothenuſe, or longeſt ſide.*

DEMONSTRATION.

In the propoſed triangle let the circle EGF be inſcribed, and from the center D, to the angular points A, B, C, and the points of contact, E, F, G, let the right-lines DA, DB, DC, DE, DF, and DG be drawn. It is plain that the ſum of the three triangles ADB, BDC, and ADC, is equal to the whole triangle ABC; but the triangle ADB is equal to the rectangle $\frac{1}{2}AB \times DG$; and ſo of the reſt; therefore the ſum of the rectangles $\frac{1}{2}AB \times DG + \frac{1}{2}CB \times DF + \frac{1}{2}AC \times DE$ is equal to the whole triangle ABC; but the ſum of theſe rectangles (*by Euc.* 1. 2.) is equal to the rectangle under half the perimeter AB + BC + AC and the ſemi-diameter DG, which laſt rectangle is, therefore, equal to the triangle given. But the angles E and G being right ones (*Euc.* 17. 3.) and the ſide AD common, and alſo DE equal to DG, thence will AE = AG (*Euc.* 47. 1.) And in the ſame manner will CE = CF; conſequently AC (AE + CE) will be = AG + CF; whence it appears



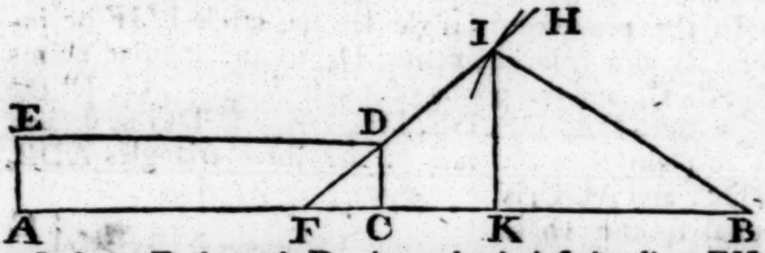
appears that the hypothenuſe is leſs than the ſum of the two legs by BG + BF, or twice the radius of the inſcribed circle, and therefore leſs than half the perimeter by once that radius, or DG: whence the propoſition is manifeſt.

PROBLEM XXXV.

The perimeter and area of a right-angled triangle being given, to deſcribe the triangle.

CONSTRUCTION.

Make AB equal to the given perimeter, which biſect in C, and upon AC let a rectangle ACDE be conſtituted equal to the given area; take CF = CD



and, from F through D, draw the indefinite line FH, to which, from B, apply BI = AF; then, upon AB let fall the perpendicular IK, ſo ſhall BIK be the triangle that was to be conſtructed.

DEMONSTRATION.

Since *(by conſtruction)* CD is = CF, therefore is IK = FK, and conſequently IK + IB + BK = FK + AF + BK = AB. Again, the exceſs of the half perimeter AC above the hyothenuſe BI (AF) being = CF = CD, it is evident *(from the premiſed lemma)* that the area of the triangle will be = ACDE = the given area by conſtruction. Q. E. D.

Method of calculation.

Dividing the area by half the perimeter, CD (= CF) will be given; then, in the triangle BFI, will be given BF, BI, and the angle F (= 45°); whence the angle B will alſo be known, and from thence BK and BI.

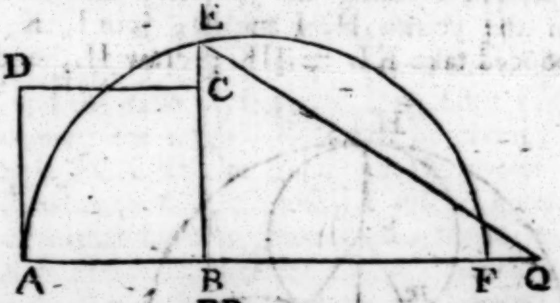
PRO-

PROBLEM XXXVI.

To make a right-angled triangle equal to a given ſquare ABCD, *whoſe ſides ſhall be in arithmetical progreſſion.*

CONSTRUCTION.

In AB produced, take $BF = \frac{3AB}{2}$, and upon AF deſcribe the ſemi-circle AEF, cutting BC produced in E; take $BQ = \frac{4EB}{3}$; join E, Q, and the thing is done.



DEMONSTRATION.

Since, by conſtruction, $QB : BE :: 4 : 3$, therefore will $BQ^2 : BE^2 :: 16 : 9$, and $BQ^2 + BE^2 : BE^2 :: 16 + 9\ (25) : 9$, that is, $EQ^2 : BE^2 :: 25 : 9$ (*Euc.* 47. 1.); whence $EQ : BE :: 5 : 3$ (*Euc.* 22. 6.) therefore the ſides BE, BQ and EQ, being to one another in the ratio of the numbers 3, 4, and 5, are in arithmetical progreſſion. And, becauſe BQ is $= \frac{4EB}{3}$, thence will

$$\frac{EB \times BQ}{2} = \frac{2EB^2}{3} = \frac{2BF \times AB}{3} = AB^2.$$ Q. E. D.

Method of calculation.

Seeing BF is $= \frac{3AB}{2}$, $(BE\sqrt{AB \times BF})$ will be $= AB\sqrt{\frac{3}{2}}$; whence BQ $(\frac{4BE}{3})$ and EQ $(\frac{5BE}{3})$ will be likewiſe given.

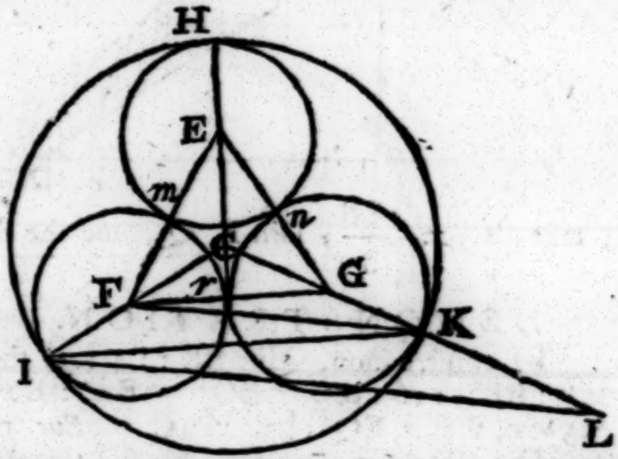
PRO-

PROBLEM XXXVII.

In a given circle CIHK, *to deſcribe three equal circles* E, F *and* G, *which ſhall touch one another, and alſo the periphery of the given circle.*

CONSTRUCTION.

From the center C let the right lines CH, CI and CK be drawn, dividing the periphery into three equal parts, in the points H, I and K; join I, K, and in CK produced take $KL = \frac{1}{2}IK$; draw IL, and parallel thereto draw KF meeting CI in F; make HE and KG each = IF, and upon the centers F, E and G, through the points, I, H and K, let the circles F*r*I, E*m*H, and G*n*K be deſcribed, and the thing is done.



DEMONSTRATION.

Draw FE, FG and EG.

Becauſe (*by conſtruction*) HE, IF, and KG are equal, CE, CF, and CG will likewiſe be equal, and FG parallel to IK (*by Euc.* 2. 6.) and therefore, KF being parallel to IL (*by conſtruction*) the triangles IKL and FGK are equiangular; whence, IK being = 2KL, FG is = 2GK (2F*r*) (*Euc.* 4. 6.) whence it is maniſeſt that the circles F and G touch each other.

Moreover, the angles ECF, ECG and FCG, as well as the containing ſides CE, CF, and CG being equal, EF, FG, and EG muſt alſo be equal (*by Euc.* 4. 1.) and therefore EF or EG = 2FI or 2GK; whence it is evident that the circles E, F and E, G alſo touch one another. But all theſe circles touch the given circle, becauſe they paſs through given points H, I, K in its periphery, and have their centers in right lines joining the center C and the points of concourſe.

Method of calculation.

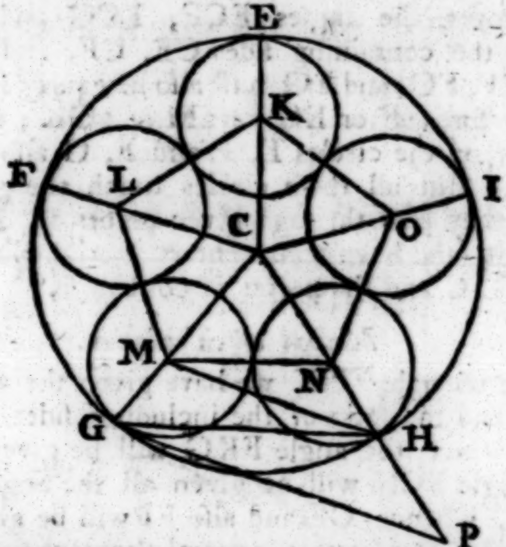
In the triangle FGK we have given the angle FGK (150^{c}) and the ratio of the including ſides (*viz.* as 2 to 1) whence the angle FKG will be given; then in the triangle FCK will be given all the angles and the ſide CK, whence CF and alſo FI will be given. But, if you had rather have a general theorem for expreſſing the ratio of FI to CI, then let EC be produced to meet FG in *r*. Therefore, the angle *r*FC being = 30°, C*r* will be = $\frac{1}{2}$CF; whence (*by Euc.* 47. 1.) FI or F*r* ($\sqrt{FC^2 - Cr^2}$) is = FC × $\sqrt{\frac{3}{4}}$, and therefore CI = FC + FC$\sqrt{\frac{3}{4}}$; conſequently CI : FC :: 1 + $\sqrt{\frac{3}{4}}$: 1: whence, by diviſion, CI : FI (:: 1 + $\sqrt{\frac{3}{4}}$: $\sqrt{\frac{3}{4}}$) :: $\sqrt{\frac{4}{3}}$ + 1 : 1.

PROBLEM XXXVIII.

In a given circle CEHG *to deſcribe five equal circles* K, L, M, N *and* O, *which ſhall touch one another, and the circle given.*

CONSTRUCTION.

Let the whole periphery EGH be divided into five equal parts, at the points E, F, G, H, and I (*by Euc.* 11. 2.) and draw CE, CF, CG, CH and CI; join G, H, and in CH produced take HP = $\frac{1}{2}$GH; draw PG, and parallel thereto draw HM, meeting CG in M; take FL, EK, IO and HN, each equal to MG, and upon the centers K, L, M, N and O, let circles be deſcribed through the points E, F, G, H and I, and the thing is done.



The demonſtration whereof is evident from the laſt propoſition: and in the ſame manner may 6, 8, or 10, *&c.* equal circles be inſcribed in a given circle, to touch one another.

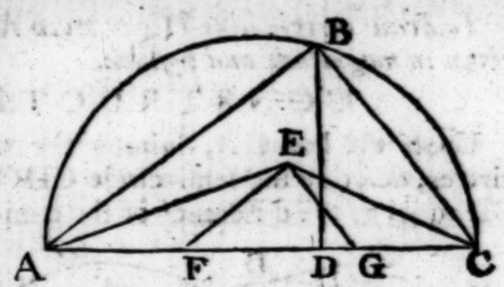
The method of calculation in this, or any other caſe, will alſo be the ſame as in the laſt problem; for in the triangle MNH will be given the ratio of NM to NH (as 2 to 1) and the included angle MNH equal to 126°, 120°, $112\frac{1}{2}$°, or 108°, *&c.* according as the number of circles is 5, 6, 8, or 10, *&c.* from which the angle MHN will be given; then in the triangle CMH will be given all the angles, and the ſide CH, to find CM.

PROBLEM XXXIX.

The perimeter of a right-angled triangle, whoſe ſides are in geometrical progreſſion, being given, to deſcribe the triangle.

CONSTRUCTION.

Upon AC, equal to the given perimeter, deſcribe the ſemi-circle ABC, and let AC be divided in D, according to extreme and mean proportion; make DB perpendicular to AC, meeting the periphery of the circle in B, and having joined A, B and C, B, let AE and CE be drawn to biſect the angles BAC, BCA; and, from point of interſection E, let EF and EG be drawn parallel to BA and BC, cutting AC in F and G; then will EFG be the triangle that was to be conſtructed.



DEMONSTRATION.

Since *(by conſtruction)* AC : AD : : AD : DC, therefore is ACq : AC × AD : : AC × AD : AC × DC *(by Euc.* 1. 6.) or ACq : ABq : : ABq : BCq *(by Cor. to Euc.* 8. 6.) and conſequently AC : AB : : AB : BC; whence, the triangles ABC, FEG, being equiangular, FG : FE : : FE : EG. Alſo EF is = AF, becauſe the angle FEA (= EAB) = FAE; and in the very ſame manner is EG = GC; therefore EF + FG + EG (= AF + FG + GC) = AC. Moreover the angle FEG (=ABC) is a right-angle, *by Euc.* 31. 3. *Q. E. D.*

Method of calculation.

Becauſe *(by conſtruction)* AD ($= \sqrt{\tfrac{5}{4}ACq} - \tfrac{1}{2}AC$) = AC × $\overline{\sqrt{\tfrac{5}{4}} - \tfrac{1}{2}}$, thence is AB ($\sqrt{AC \times AD}$) = AC × $\sqrt{\sqrt{\tfrac{5}{4}} - \tfrac{1}{2}}$, and BC ($\sqrt{AC \times CD}$ = AD) = AC × $\overline{\sqrt{\tfrac{5}{4}} - \tfrac{1}{2}}$: but, by reaſon of the ſimilar triangles ABC, FEG, it will be as AC + AB + BC : (FG + FE + EG) AC : : AC : FG : : AB : FE : : BC : EG; or as $\sqrt{\sqrt{\tfrac{5}{4}} - \tfrac{1}{2}} + \tfrac{1}{2} + \sqrt{\tfrac{5}{4}}$: 1 : : AC : FG : : AB : FE : : BC : EG; whence FG, FE, and EG are given.

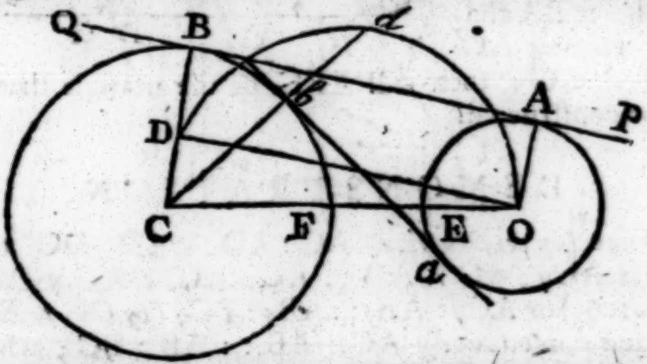
PRO-

PROBLEM XL.

To draw a right-line PQ to touch two circles C and O, given in magnitude and poſition.

CONSTRUCTION.

Upon the line CO, joining the centers of the given circles, deſcribe the ſemi-circle CDO, in which inſcribe CD equal to the difference of the ſemi-diameters CF and OE; and from the point B, where CD produced meets the periphery BF, draw PB perpendicular to CB; then will BP touch both the circles.



DEMONSTRATION.

Join O, D, and draw OA perpendicular to PQ.

The angle CDO, ſtanding in a ſemi-circle, is right; therefore, the angles B and A being both right ones, by conſtruction, the angle AOD muſt alſo be right, and the figure DOAB a rectangle, and conſequently $AO = BD = BC - CD = CF - CD = OE$ *(by conſtruction)*. Wherefore, ſeeing CB and OA are reſpectively equal to CF and OE, and both the angles A and B right ones, it is evident that the right line PQ touches both the circles. Q. E. D.

The numerical ſolution of this problem is extremely eaſy; for, ſince the two ſides CO and CD of the right-angled triangle CDO are both given, the angles DCO and AOC, determining the points of contact B and A, are from thence given, at one operation.

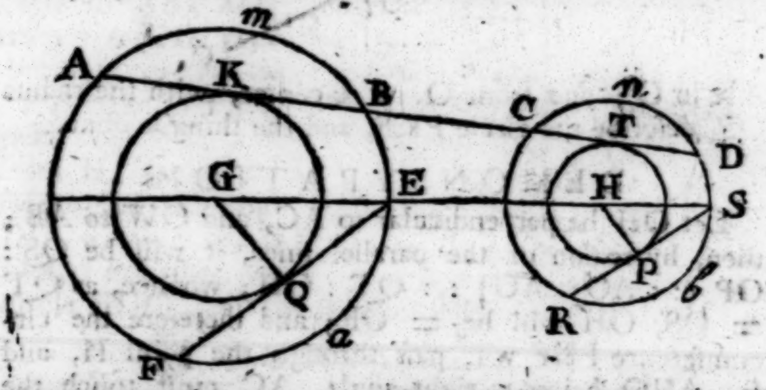
But if it be required to draw a right-line *(ab)* to touch both circles, and to paſs between the centers C and O; then, inſtead of taking CD equal to the difference of the ſemi-diameters CF, OE, let C*d* be taken equal to their ſum, and the reſt of the proceſs will be exactly the ſame.

PROBLEM XLI.

To draw a right-line AD *through two circles* GAEF, HCSR, *given in magnitude and poſition, ſo as to cut off ſegments thereof,* AKB*m*, CTD*n*, *equal reſpectively to two given ſegments* EQF*a*, SPR*b*.

CONSTRUCTION.

Upon the ſubtenſes EF, SR, from the centers G and H, let fall the perpendiculars GQ and HP; and from the ſame centers, at the diſtances GQ, HP, let two circles GQK, HPT be deſcribed; then draw a right-line AD to touch both theſe circles, by the laſt propoſition, and the thing is done; for the lines FE, AB being at the ſame diſtance from the center G, the ſegments cut off by them muſt conſequently be equal: and, in like manner, the ſegments SPR*b*, CTD*n*, are alſo equal.

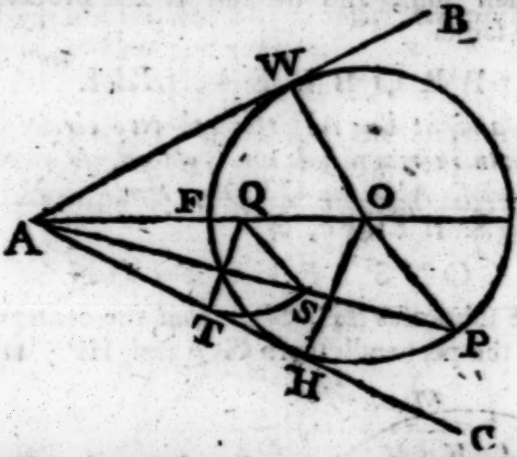


PROBLEM XLII.

To deſcribe the circumference of a circle through a given point P, *to touch two right-lines* AB, AC, *given by poſition.*

CONSTRUCTION.

Join A, P, and biſect the angle BAC, with the right-line AK, and, from any point Q in that line, draw QT perpendicular to AC; then, from Q to AP, draw QS = QT; draw likewiſe PO parallel to SQ, meeting



AK in O; and from O, as a center, with the radius OP, deſcribe the circle PKF, and the thing is done.

DEMONSTRATION.

Let OH be perpendicular to AC, and OW to AB; then, by reaſon of the parallel lines, it will be QS : OP (: : AQ : AO) : : QT : OH; whence, as QT = QS, OH will be = OP; and therefore the circumference PKF will paſs through the point H, and ſo, AHO being a right-angle, AC muſt touch the circle in that point. Moreover, the triangles AOH and AOW being equiangular and having one ſide common, OW will therefore be = OH, and the circle alſo touch AB in the point W. *Q. E. D.*

Method of calculation.

Having aſſumed AQ at pleaſure, there will be given, in the triangle AQT, all the angles and one ſide, whence QT (= QS) will be given: then, in the triangle AQS, will be given AQ, QS, and the angle QAS, whence the angle AQS (= AOP) will be given. Laſtly, in the triangle AOP will be given all the angles and the ſide AP, whence AO and PO will be given.

Otherwiſe.

Say, as the ſine of OAH : radius (: : OH : OA : :

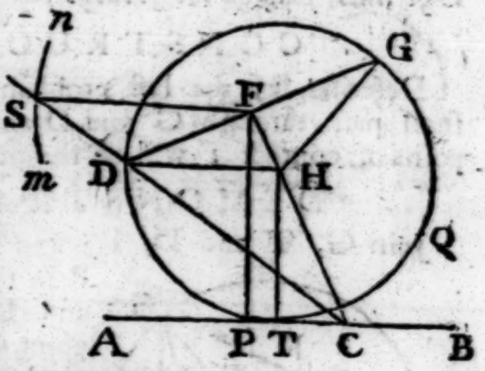
OP : OA) : : the ſine of OAP : ſine of OPA; then, in the triangle AOP will be given all the angles and the ſide AP, whence the other ſides AO and OP will be found.

PROBLEM XLIII.

To deſcribe the circumference of a circle through two given points D, G, *to touch a right-line* AB, *given by poſition.*

CONSTRUCTION.

Draw DG, and biſect the ſame by the perpendicular FC, meeting AB in C; join C, D, and make FP perpendicular to AB; and, from F to CD, produced, draw FS = FP; make DH parallel to FS, and from H, the interſection of CF and DH, with the radius DH, deſcribe the circle HDQ, and the thing is done.



DEMONSTRATION.

Join H, G, and draw HT parallel to FP, meeting AB in T; then becauſe of the parallel lines, it will be, FS : HD (: : CF : CH) : : FP : HT; wherefore, as the antecedents FS and FP are equal, the conſequents HD and HT muſt likewiſe be equal; and therefore ſince HT is perpendicular to AB, the circumference of the circle will touch AB in T; and it will alſo paſs through the point G, becauſe the two triangles DFH, GFH, having two ſides and the included angles equal, are equal in every reſpect. Q. E. D.

Method of calculation.

The angle FCA, and the numbers expreſſing FC and DG being given, in the triangle CFD will be given (beſides the right-angle) both the legs CF and FD, whence CD and the angle FCD will be known;

then it will be, as the ſine of FCA (TCH) : radius (: : TH : CH : : DH : CH) : : the ſine of HCD : the ſine CDH; therefore in the triangle HCD there will be given all the angles and the ſide CD, whence CH and HD will be known.

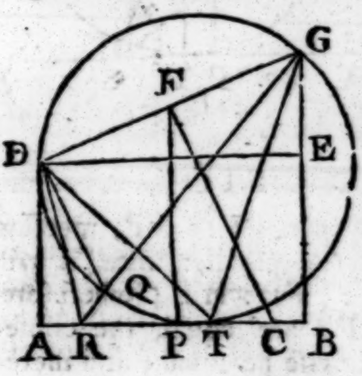
PROBLEM XLIV.

Having given AB, *and alſo* AD *and* BG, *perpendicular to* AB; *to find a point* T *in* AB, *to which if two right-lines* DT, GT *be drawn, the angle* DTG, *formed by thoſe lines, ſhall be the greateſt poſſible.*

CONSTRUCTION.

Deſcribe, by the laſt problem, a circle GDQ, that ſhall paſs through G and D and touch AB, and the point of contact T will be the point required.

DEMONSTRATION.



Join G, T and D, T; and from any other point R in the line AB, draw RG and RD; alſo, from the point Q where GR cuts the circle, draw QD: then, the angle GQD, being external with regard to the triangle DQR, will be greater than GRD; therefore GTD, ſtanding in the ſame ſegment with GQD, will be alſo greater than GRD. *Q. E. D.*

Method of calculation.

Draw DE parallel to AB; then in the triangle GDE will be given DE, EG (= BG — AD) and the right-angle DEG, whence the other angles EDG, EGD, and the ſide DG will be found; then in the triangle CFP, ſimilar to GDE, will be given all the angles and the ſide FP $(=\frac{AD+BG}{2})$ whence FC will be given; from which, by proceeding as in the laſt problem, all the reſt will be found.

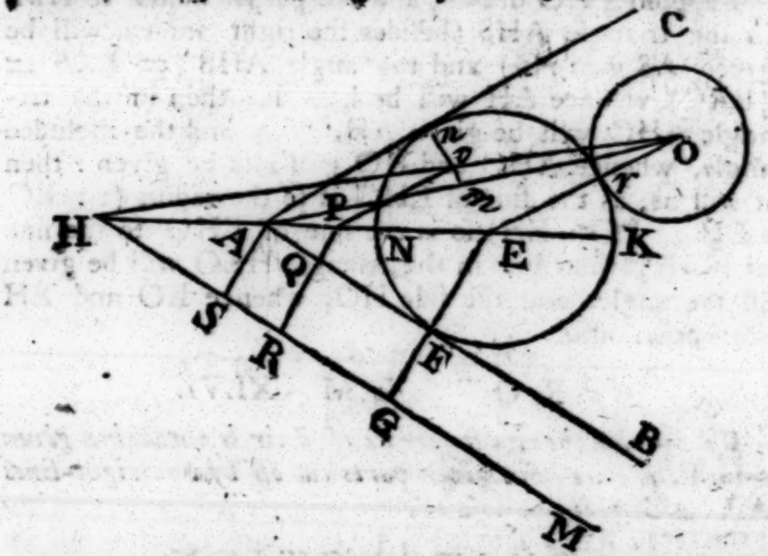
PRO-

PROBLEM XLV.

To deſcribe a circle, which ſhall touch two right-lines AB, AC, *given in poſition, and alſo another circle* O, *given in magnitude and poſition.*

CONSTRUCTION.

Let the angle CAB, made by the concourſe of the two lines, be biſected by AK; and, from any point P in this line, let fall PQ perpendicular to AB, which produce to R, ſo that QR may be equal to the ſemi-diameter of the given circle; and through R, parallel to AB, draw HM, meeting KA produced in H; draw HO, to which, from P, draw P*v* = PR, and draw OE parallel to P*v*, meeting AK in E, and cutting the periphery of the given circle in *r*; laſtly, from E, with the radius E*r*, deſcribe the circle E*r*KN, and the thing is done.



DEMONSTRATION.

Draw EG perpendicular to HM, cutting AB in F: then, by reaſon of the parallel lines, PR : EG (: : HP : HE) : : P*v* : EO; therefore PR being = P*v* *(by conſtruction)* EG and EO muſt likewiſe be equal; from which the equal quantities FG and O*r* being taken away, the remainders EF and E*r* will be equal; and therefore the circumference *r*KN paſſes through F; but it alſo touches AB in that point, becauſe EF *(by conſtruction)* is perpendicular to AB; it likewiſe touches AC, becauſe AE biſects the angle BAC; laſtly, it touches the circle O, becauſe the right-line OE joining the centers O and E, paſſes through the point *r*, common to both peripheries.

Method of calculation.

Suppoſing AO drawn, and AS perpendicular to HM, in the triangle AHS (beſides the right angle) will be given AS (= *r*O) and the angle AHS (= EAF = ½BAC) whence AH will be known; then in the triangle AHO will be given AH, AO, and the included angle, whence AHO and HO will alſo be given: then it will be, as the ſine of EHG is to the radius (: : EG : EH : : EO : EH) ſo is the ſine of EHO to the ſine of EOH; therefore in the triangle HEO will be given all the angles and the ſide HO, whence EO and EH are known alſo.

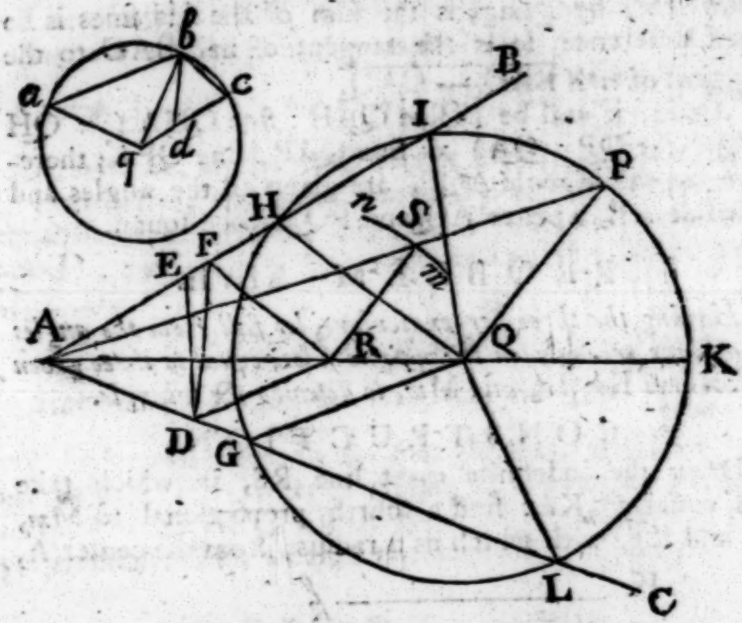
PROBLEM XLVI.

To deſcribe the circumference of a circle through a given point P, *ſo as to have given parts cut off by two right-lines* AB, AC *given in poſition.*

CONSTRUCTION.

Let the arcs to be cut off by AC and AB be ſimilar reſpectively to the arcs *ab*, *bc*, of any given circle *abcq*, whoſe chords *ab*, *bc* ſubtend, at the center, any given angles *aqb*, *bqc*. Let the angle *abc* be biſected by *bd*; take, in AB and AC any two points, E, D, equidiſtant from A; and, having drawn DE, make the angle EDF = *qbd*, CDR = *qba*, and BFR = *qbc*; then from the interſection R of the lines DR and FR, with the radius RD, deſcribe an arch *m*S*n*, cutting the line AP in S, draw RS and ARK, and alſo PQ, parallel to RS, meeting AK in Q; then from the center Q, with the radius PQ, deſcribe the circle KPI, and the thing is done.

DEMON-



DEMONSTRATION.

Draw QH and QG parallel to RF and RD, meeting AB and AC in H and G. The angles BED and CDE being equal, BFD will exceed CDF by twice EDF, or by twice *qbd*, that is, by as much as *qbc* exceeds *qba*, or laſtly, by as much as BFR exceeds CDR; therefore, ſeeing the whole angle BFD as much exceeds the whole angle CDF, as the part BFR of the former exceeds the part CDR of the latter, the remaining parts RFD and RDF muſt be equal, and conſequently FR = RD = RS. But by reaſon of the parallel lines it will be, RF : QH : : RD : QG : : RS : QP; whence, the antecedents RF, RD, RS, being equal, the conſequents QH, QG, QP, muſt be equal too, and the circumference paſs through the points H and G; whence the ſolution is manifeſt.

Method of calculation.

If two perpendiculars be conceived to fall from Q upon AB and AC, they will, it is plain, be in the given ratio of the ſines of the angles QHI and QGL; therefore the poſition of the line AQK will be given (*from*

Prob. 26.) by ſaying, as the ſum of the ſaid ſines is to their difference, ſo is the tangent of half BAC to the tangent of half $\overline{\text{BAQ} - \text{CAQ}}$.

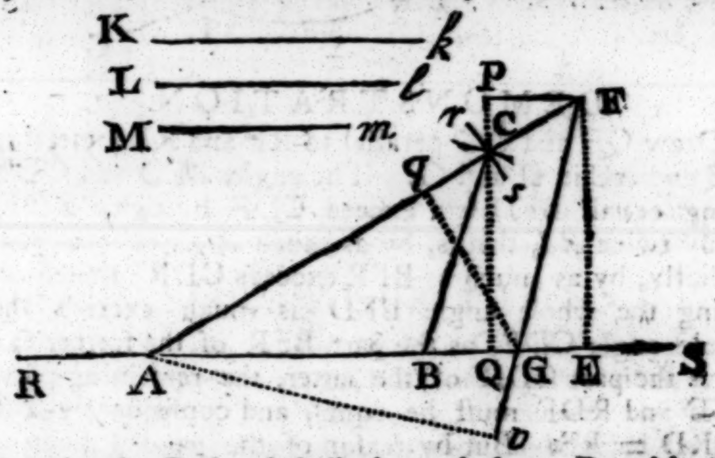
Again, it will be as ſine QAH : ſine QHA (: : QH : QA : : QP : QA) : : ſine QAP : ſine QPA; therefore, in the triangle AQP, are given all the angles and one ſide AP, whence AQ and PQ will be found.

PROBLEM XLVII.

Having the three perpendiculars, let fall from the angles of a plane triangle on the oppoſite ſides, equal to three given right-lines K*k*, L*l*, *and* M*m*, *to deſcribe the triangle.*

CONSTRUCTION.

Draw the indefinite right line RS, in which take AB equal to K*k*: find a fourth proportional to M*m*, L*l*, and K*k*, with which as a radius, from the center A, let an arch *r*C*s* be deſcribed; and from B, with the radius L*l*, let another arch be deſcribed interſecting the former in C; join A, C and B, C, and upon RS let fall the perpendicular QC, in which produced, take QP = L*l*, and draw PF parallel to RS, meeting AC, produced, in F, draw FG parallel to CB, and AFG will be the triangle required.



DEMONSTRATION.

Draw FE, G*q* and A*v*, perpendicular to the three ſides of the triangle.

The triangles ABC, AGF; AFE, AGq; and GFE AGv, are equi-angular, *by conſtruction*; therefore Gq : FE : : AG : AF : : AB (Kk) : AC ($\frac{Kk \times Ll}{Mm}$) : : Mm : Ll; whence, as the conſequents FE and Ll are equal, *by conſtruction*, the antecedents Gq and Mm muſt be equal likewiſe. Again, BC (Ll) : AB (Kk) (: : FG : AG) : : FE (Ll) : Av; and conſequently Kk = Av. Q. E. D.

Method of calculation.

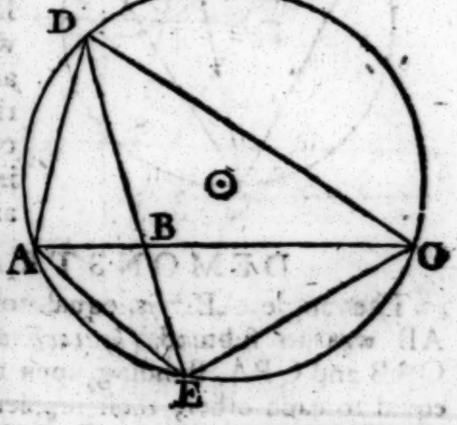
Since Kk, Ll, and Mm are given, AC ($= \frac{Kk \times Ll}{Mm}$) will be known; then in the triangle ABC will be given all the three ſides, whence the angles are known; laſtly, in the triangle AFG will be given all the angles and the perpendicular EF, whence the ſides are alſo known.

PROBLEM XLVIII.

The poſition of three points, in the ſame right-line, being given, it is propoſed to find a fourth, where lines, drawn from the former three, ſhall make given angles with each other.

CONSTRUCTION.

Let the three given points be A, B, and C: make the angles ACE and CAE reſpectively equal to the given angles which the lines drawn from B, A, and B, C are to make; and let AE and CE meet in E; thro' A, C and E, let the circumference of a circle AECD be deſcribed, and, thro' E and B, draw EBD, meeting it in D, then will D be the point required.



DEMON-

DEMONSTRATION.

Join A, D, and C, D.

The angle EDA is equal to ACE, ſtanding on the ſame ſegment; and, for the like reaſon, is EDC = CAE. Q. E. D.

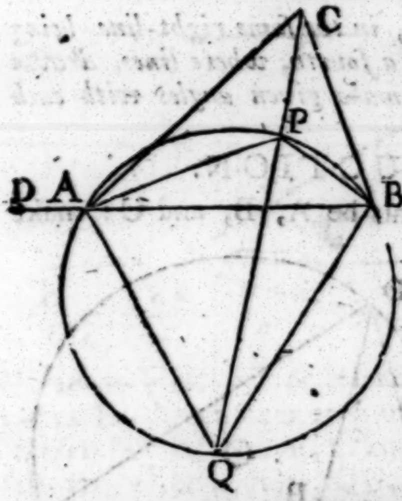
Method of calculation.

In the triangle ACE are given all the angles and the ſide AC, whence AE will be given; then, in the triangle ABE, will be given the two ſides AE, AB, and the included angle, whence ABE and all the reſt of the angles in the figure will be given.

PROBLEM XLIX.

Three points A, B, C, *being, any how, given; to find a fourth, where lines, drawn from the former three, ſhall make given angles with one another.*

CONSTRUCTION.



Join the given points, and upon the right-line AB deſcribe a ſegment of a circle, capable of the given angle which that line is to ſubtend; compleat the circle, produce BA, and make the angle DAQ equal to the angle which BC is to ſubtend, and let AQ meet the periphery in Q; draw QC, cutting the ſame periphery in P; join A, P, and B, P, and the thing is done.

DEMONSTRATION.

The angle ABP is equal to the given angle which AB was to ſubtend (*by conſtruction*); and the angles QAB and QPA, ſtanding upon the ſame ſegment, being equal to each other, their ſupplements DAQ and BPC muſt likewiſe be equal. Q. E. D.

Method of calculation.

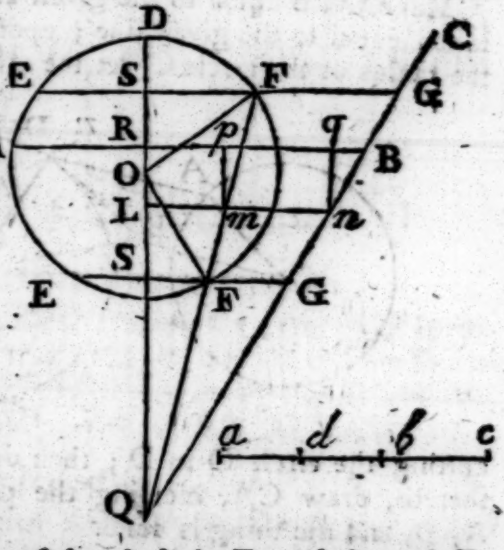
Join B, Q; then, in the triangle ABQ will be given all the angles and the ſide AB, whence BQ and ABQ will be known; then in the triangle CBQ will be given two ſides, and the included angle CBQ, whence the angle CQB, equal to BAP, will be known: laſtly, in the triangle APB will be given all the angles and the ſide AB, from which AP and BP will be found.

PROBLEM L.

To draw a right-line EG *through a circle* O, *given in magnitude and poſition, which ſhall alſo cut a right-line* QC, *given by poſition, in a given angle, and have its parts* EF, FG, *intercepted by the circle and that right-line, in the given ratio of the two right-lines ab and bc.*

CONSTRUCTION.

At any point B, in the right-line QC, make the angle QBA equal to the given angle, and through the center O, perpendicular to BA, draw DQ meeting BA in R, and CG in Q; biſect *ab* in *d*, and in RB take R*p* = *bd*, and *pq* = *bc*, and draw *pm* and *qn* parallel to DQ; from the point *n*, where *qn* interſects QC, draw *n*L parallel to BA, meeting *pm* in *m*; through the points Q and *m* draw Q*m*F, cutting the periphery of the circle in F, and through F, parallel to BA, draw EFG, and the thing is done.



DEMONSTRATION.

The lines GE, BA, and *n*L, being parallel, the angles QGE, QBA, *&c.* will be equal, and likewiſe SF : FG : : $Lm : mn$; but Lm *(by conſtruction)* is $(= Rp) = db$, and mn $(= pq) = bc$; therefore SF : FG : : $db : bc$, and conſequently EF (2SF) : FG : : ab $(2bd) : bc$. Q. E. D.

Method of calculation.

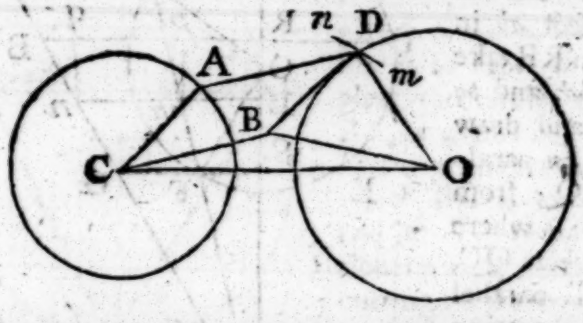
Ln (dc) : Lm (db) : : the tangent of LQn (the complement of the given angle QBR) : the tangent of LQm; therefore, in the triangle OQF, will be given one angle OQF and two ſides, QO, FO; whence, not only the angle SOF, but alſo SO and SF will be known.

PROBLEM LI.

To apply, or inſcribe, a given right-line AD *between the peripheries of two circles* C *and* O, *given in magnitude and poſition, ſo as to be inclined to the right-line* CO *joining the centers in a given angle.*

CONSTRUCTION.

Make OCB equal to the given angle, and let CB be taken equal to the given line; upon the center B, with the radius of the circle C, let the arch nDm be deſcribed, cutting the circle O in D; then draw BD, and parallel thereto, draw CA, meeting the periphery in A; join A, D, and the thing is done.



DEMONSTRATION.

Becauſe *(by conſtruction)* CA and BD are equal and parallel, therefore will AD and CB be alſo equal and parallel *(by Euc.* 33. 1.) Q. E. D.

Method of calculation.

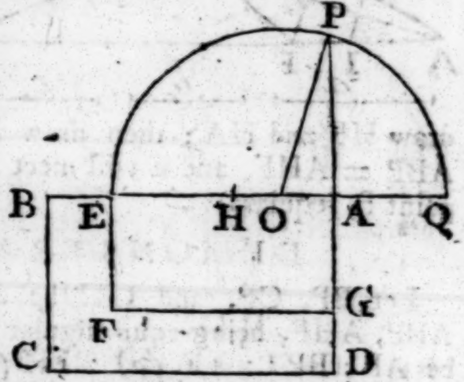
In the triangle CBO are given two ſides CO and CB, and the angle OCB, whence OB and the angle COB will be known; then in the triangle OBD will be given all the three ſides, whence all the angles, and conſequently DOC, will alſo be known.

PROBLEM LII.

From a given rectangle ABCD, *to cut off a gnomon* ECG, *whoſe breadth ſhall be every-where the ſame, and whoſe area ſhall be juſt half that of the rectangle.*

CONSTRUCTION.

In BA take BH equal to BC, or AD; and in DA produced take AP a mean-proportional between BA and ½AD (ſo that AP^2 may = the given area AGFE). From P to the middle of AH draw PO; make OE = OP, and DG = BE; compleat the rectangle EAGF, and the thing is done.



DEMONSTRATION.

If the ſemi-circle EPQ, from the center O, be deſcribed, it is plain that AQ = EH = BH — BE = AD — DG = AG; and conſequently that AE × AG = AE × AQ = AP^2 (*Euc.* 13. 6.) Q. E. D.

Method of calculation.

In the right-angled triangle AOP are given AO $\left(= \frac{AB - BC}{2}\right)$ and AP $\left(= \sqrt{\tfrac{1}{2}AB \times BC}\right)$; whence OP will be known, and from thence both AE and AG.

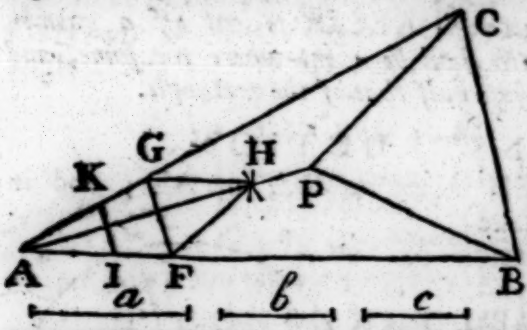
PRO-

PROBLEM LIII.

Three points A, B, C, *being given, it is propoſed to find a fourth,* P, *from whence lines, drawn to the three former, ſhall obtain the ratio of three given lines, a, b and c, reſpectively.*

CONSTRUCTION.

Having joined the given points, take AF, in AB, equal to a, and AI = c; alſo make the angles AFG and AIK equal, each, to ACB; and from the centers F and G, with the radii b and AK reſpectively, let two arcs be deſcribed interſecting in H; from which point draw HF and HA; then draw BP to make the angle ABP = AHF, and it will meet AH (produced) in the point P, required.



DEMONSTRATION.

Let BP, CP, and GH be drawn. The triangles ABP, AHF, being equi-angular (*by conſtruction*) it will be AP : BP : : AF (a) : FH (b); alſo AB : AP : : AH : AF; and AB : AC : : AG : AF (becauſe ABC and AGF are likewiſe equi-angular) whence it is evident, ſince the extremes of the two laſt proportions are the ſame, that AP × AH = AC × AG, or AC : AP : : AH : AG; therefore the triangles ACP, AHG being equi-angular (*Euc.* 6. 6.) we have AP : CP : : AG : GH (AK) : : AF (a) : AI (c). Q. E. D.

Method of calculation.

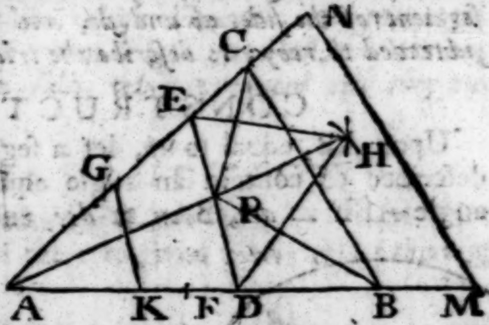
In the triangles AFG, AIK are given all the angles and the ſides AF and AI, whence AG, FG, and AK (GH) will be found; then in the triangle FGH will be given all the ſides, to find the angle HFG; which, added to AFG, gives AFH (APB) from whence, and the two given ſides AF and FH including it, every thing elſe is readily determined.

PROBLEM LIV.

To deſcribe a triangle (ABC) *ſimilar to a given one* AMN, *ſuch that three lines* (AP, BP, CP) *may be drawn from its angular points to meet in the ſame point* (P) *ſo as to be equal to three given lines* AD, AF, *and* AK, *reſpectively.*

CONSTRUCTION.

Draw DE and KG, making the angles ADE and AKG, each, equal to the given angle N, and interſecting AN in E and G; from the centers D and E, with the intervals AF and AG, let two arcs be deſcribed, interſecting in H; draw AH, in which take AP = AD; and from P, to AM and AN, apply PB and PC equal, reſpectively, to AF and AK, and let B, C be joined; ſo ſhall ABC be the triangle that was to be determined.



DEMONSTRATION.

The three lines AP, BP, CP, are, reſpectively, equal to the three given lines AD, AF, AK, by conſtruction; we therefore have only to prove that the triangle ABC is ſimilar to the given one AMN. Now ſuppoſing DH and EH to be drawn, it will be AP : PC (or AD : AK) : : AE : AG (EH); whence the triangles APC and AHE will be equi-angular (*Euc.* 6. 6.) and conſequently AC : AH : : AP (AD) : AE : : AN : AM (*Euc.* 5. 6.): but the triangles ABP and ADH (having AP = AD, PB = DH (*by conſtruction*) and the angle DAP common) are equal in all reſpects; therefore, by ſubſtituting AB in the room of AH, our laſt proportion becomes AC : AB : : AN : AM; whence it is manifeſt that the triangles ABC and AMN are equi-angular. Q. E. D.

Method of calculation.

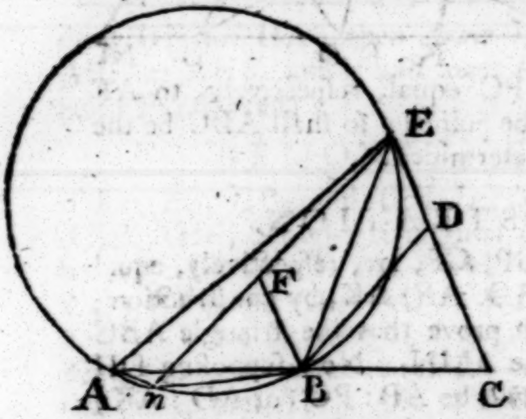
In the triangles ADE, AKG, are given all the angles and the ſides AD and AK, from which AE, DE, and AG will be known; then in the triangle DHE will be given all the ſides, to find the angle EDH, which added to ADE gives ADH; from whence, and the two given ſides including it, AH (= AB) will be known.

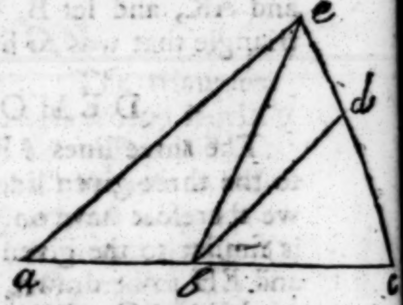
PROBLEM LV.

In the triangle ace, beſides the angle c, are given the ſegments of the ſides ab and de, and the angles aeb and dbe ſubtended thereby; to deſcribe the triangle.

CONSTRUCTION.

Upon AB, equal to *ab*, let a ſegment of a circle be deſcribed to contain an angle equal to *aeb*; make the angle ABF = *ace*, BA*n* = *dbe*, and the line BF = *ed*; from the point *n*, where A*n* cuts the periphery of the circle, through F, draw *n*FE, meeting the periphery in E; join A, E, and B, E, and draw EC parallel to BF, meeting AB, produced, in C; and then the thing is done.





DEMONSTRATION.

Let BD be parallel to FE.

Since the lines BD, EF, and ED, FB, are parallel, therefore is ED = BF (= *ed*), and the angle ACE alſo = ABF (*ace*) *Euc.* 28. 1. Moreover, the angle BE*n* (DBE)

(DBE) is equal to BA*n* (*dbe*), both ſtanding upon the ſame ſegment B*n*. Q. E. D.

Method of calculation.

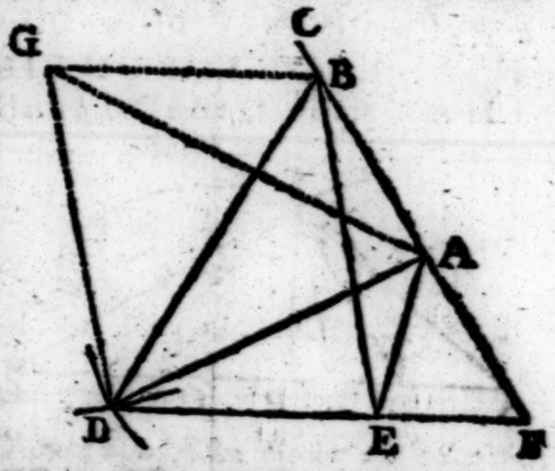
Join B, *n*; then in the triangle AB*n* will be given all the angles and the ſide AB, whence B*n* will be known; then in the triangle *n*BF will be given B*n*, BF, and the included angle *n*BF, whence BF*n* (CDB) and all the reſt of the angles in the figure will be known.

PROBLEM LVI.

To make a trapezium, whoſe diagonals, and two oppoſite ſides, ſhall be all of given lengths, and whereof the angle formed by the given ſides, when produced till they meet, ſhall alſo be given.

CONSTRUCTION.

Draw the indefinite right-line AC, and take therein AB equal to one of the two given ſides; make the angle CBG equal to the given angle, and let BG be made equal to the other given ſide; upon the centers A and G, with intervals equal to the two diagonals, let two arches be deſcribed, cutting each other in D; make DE equal, and parallel, to GB; join D, B, and E, A; then ABDE will be the trapezium required.



DEMONSTRATION.

Draw DG, DA and BE, and let BA and DE be produced to meet each other in F.

The lines BG and DE are equal, and parallel, by conſtruction; therefore BE is = DG, which laſt (*by* *conſtruction*) is equal to one of the given diagonals, as AD is equal to the other: moreover, the ſides AB and ED (BG) are equal to the given ſides, by conſtruction; and the angle F is equal to the given angle CBG, becauſe DF is parallel to GB. Q. E. D.

Method of calculation.

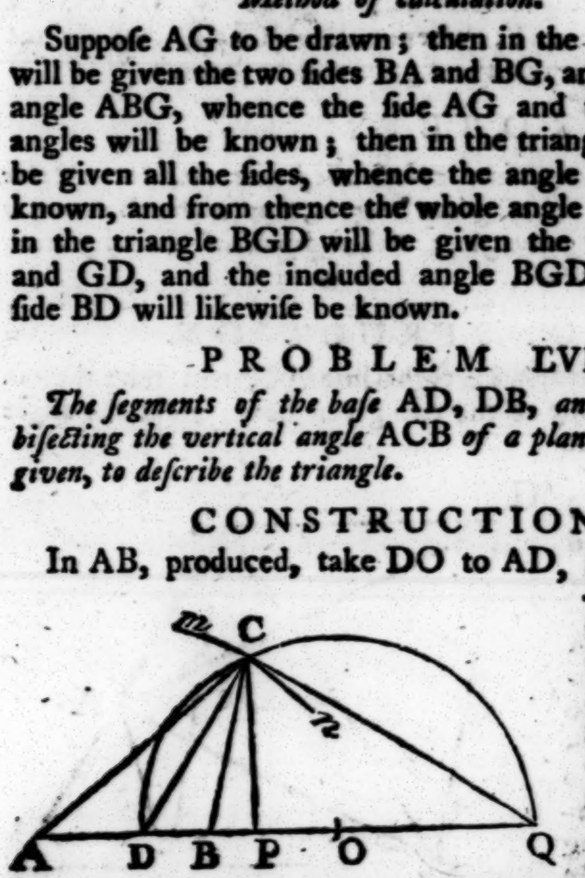
Suppoſe AG to be drawn; then in the triangle ABG will be given the two ſides BA and BG, and the included angle ABG, whence the ſide AG and the other two angles will be known; then in the triangle ADG will be given all the ſides, whence the angle AGD will be known, and from thence the whole angle BGD; laſtly, in the triangle BGD will be given the two ſides BG and GD, and the included angle BGD, whence the ſide BD will likewiſe be known.

PROBLEM LVII.

The ſegments of the baſe AD, DB, *and the line* DC *biſecting the vertical angle* ACB *of a plane triangle being given, to deſcribe the triangle.*

CONSTRUCTION.

In AB, produced, take DO to AD, as DB to AD — DB, and from the center O, with the radius OD, deſcribe the circle DCQ; alſo from the center D, at the given diſtance DC, deſcribe the circle *m*C*n*, and from C, the interſection of the two circles, draw CA and CB, and the thing is done.



DEMONSTRATION.

Since DO : AD : : DB : AD — DB; therefore (*by the lemma in p.* 334.) AC : CB : : AD : DB: whence CD biſects the angle ACB (*by Euc.* 3. 6.) Q. E. D.

Method of calculation.

Draw CP perpendicular to AQ.

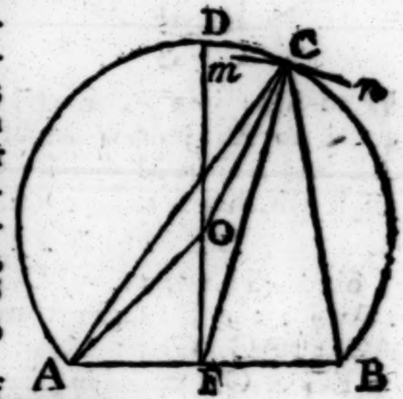
Becauſe, by conſtruction, OD is $= \frac{AD \times BD}{AD - BD}$, therefore will $DQ = \frac{2AD \times BD}{AD - BD}$; whence, by reaſon of the ſimilar triangles DCQ, DPC, it will be, as $\frac{2AD \times BD}{AD - BD} : DC :: DC : DP = \frac{\overline{AD - BD} \times DC^2}{2AD \times BD}$; whence AC and CB are given.

PROBLEM LVIII.

Having given the baſe, the angle at the vertex, and the line drawn from thence to biſect the baſe; to conſtruct the triangle.

CONSTRUCTION.

Upon the given baſe AB deſcribe (*by Prob.* 4.) a ſegment of a circle ADB capable of the given angle; and, from the point F, in which the perpendicular DF biſects AB, with a radius FC equal to the biſecting line, deſcribe *n*C*m*, cutting the periphery ACB in C; join A, C and B, C, and the thing is done.



The demonſtration of which is evident from the conſtruction.

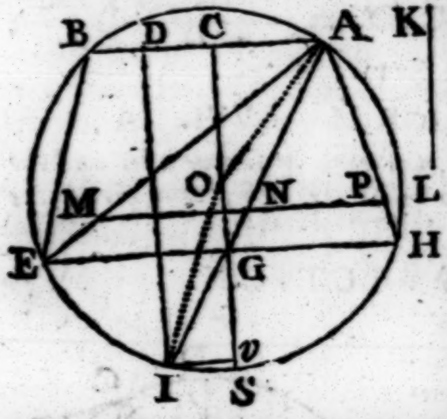
Method of calculation.

From the center O let OA and OC be drawn; then in the triangle AOF will be given all the angles and the ſide AF, whence FO and OC (OA) will be known; and in the triangle CFO will be given all the ſides, whence the angle FOC, and its ſupplement DOC, expreſſing the difference of the angles at the baſe, will alſo be known.

PROBLEM LIX.

The baſe, the difference of the angles at the baſe, and the line drawn from the vertical angle to biſect the baſe of any plane triangle, being given; to deſcribe the triangle.

CONSTRUCTION.



Upon AB, equal to the given baſe, let a ſegment of a circle AHEB be deſcribed to contain an angle equal to the difference of the angles at the baſe; biſect AB in C, and take CD to AC in the duplicate ratio of AC to the given biſecting line KL; make CS and DI perpendicular to AB, cutting the circle in S and I; draw AI, cutting CS in G; and, through G, draw the chord EGH parallel to AB; join A, E and A, H, and in AI take AN equal to KL; draw MNP parallel to EH, meeting AE and AH in M and P; then will AMP be the triangle which was to be conſtructed.

DEMONSTRATION.

Since, *(by conſtruction)* CG is parallel to DI, and KL*q* : AC*q* : : AC : CD; therefore *(Euc.* 4. 6.) KL*q* : AC*q* : : AG : GI : : AG*q* : GI × AG: but GI × AG = EG × GH = EG*q* *(Euc.* 35. 3. *and* 3. 3.) therefore KL*q* : AC*q* : : AG*q* : EG*q*; and conſequently KL : AC : : AG : EG : : AN : NM; but AN is *(by conſtruction)* equal to KL, therefore NM is = AC, and conſequently MP (2MN) = AB. Moreover the difference of the angles at the baſe, P — M, is (= AHE — AEH) = AEB; which *(by conſtruction)* is equal to the difference given. Q. E. D.

Method of calculation.

From the center O draw OA and OI, alſo draw I*v* parallel to EH, meeting OS in *v*: then it will be *(by con-*

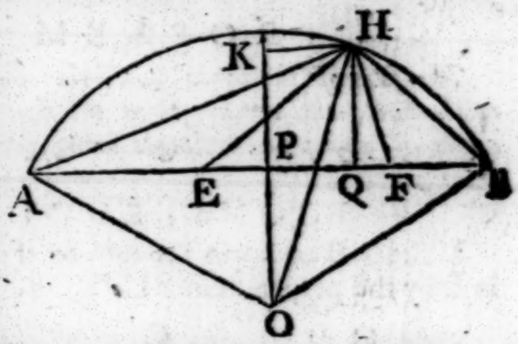
conſtruction) as KL*q* : AC*q* (: : AC : I*v*) : : the ſine of AOC or AEB, the given difference of the angles at the baſe, to the ſine of SOI; which, added to AOS, gives AOI, whoſe ſupplement, divided by 2, will be OIG; from whence OGI and its ſupplement OGA are given, and conſequently ANM (equal to AGE); then in the triangle ANM will be given AN, NM, and the included angle ANM, whence the angles M, A, P, will alſo be given.

PROBLEM LX.

The perpendicular, the angle at the vertex, and the ſum of the three ſides of a triangle being given; to deſcribe the triangle.

CONSTRUCTION.

Make AB equal to the ſum of the ſides, which biſect in P, making PO perpendicular to AB, and the angle PAO equal to half the given angle at the vertex; from the center O with the radius OA deſcribe the circle AHB, and in OP, produced, take PK equal to the given perpendicular, and draw KH parallel to BA, cutting the circle in H; join A, H and B, H, and make the angles BHF and AHE equal to HBF and HAE reſpectively; then will EHF be the triangle required.



DEMONSTRATION.

Join O, B and O, H, and draw HQ perpendicular to AB.

The angle EFH is = BHF + HBF = 2HBF *(by conſtruction)* = HOA *(Euc.* 20. 3.) : and, in the ſame manner is FEH = HOB; hence it follows that EFH + FEH (= HOA + HOB) = AOB; and, by taking each of theſe equal quantities from two right-angles, we have $EHF = OAB + OBA$ (*Euc.* 32. 1.) $= 2OAB$ = the given angle (*by conſtruction*). Moreover QH is $= PK$ = the given perpendicular; and EH being $= AE$, and $FH = BF$ (*Euc.* 6. 1.), $EH + HF + EF$ will therefore be $= AB$ = the given ſum of the ſides. Q. E. D.

Method of calculation.

In the triangle AOP are given all the angles and the ſide AP, whence OP and AO (HO) are known; then in the triangle OHK will be given the ſides OH and OK ($OP + PK$) whence HK will be given; next, in the triangle BQH will be given QH and BQ ($BP - HK$) whence QBH, and its double QFH, will be given; laſtly, in the triangle EFH are given all the angles and the perpendicular QH, whence the ſides will alſo be given.

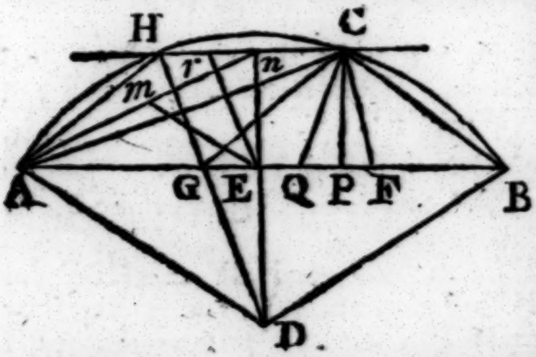
But the anſwer may be more eaſily brought out, by firſt finding, HOK, the difference of the angles ABH and BAH, as in the fifth Problem.

PROBLEM LXI.

The ſum of the three ſides, the difference of the angles at the baſe, and the length of the line biſecting the vertical angle of any plane triangle being given; to deſcribe the triangle.

CONSTRUCTION.

Make AB equal to the ſum of the ſides, which biſect in E by the perpendicular DE*n*, and make the angle *n*E*r* equal to half the given difference of the angles at the baſe; taking E*r* equal to the line biſecting the vertical angle; through *r* draw C*nr* parallel to AB, cutting DE*n* in *n*; draw *n*A, to which draw E*m* = E*r*, and draw AD parallel to E*m*, meeting *n*ED in D; and on the center D, at the diſtance of



of DA, deſcribe the circle ACB, cutting *rn*C in C; join A, C and B, C, and make the angle BCF = CBF, alſo make ACG = CAG, and let CF and CG meet AB in F and G; then will FCG be the triangle that was to be deſcribed.

DEMONSTRATION.

Upon AB let fall the perpendicular CP; let CQ biſect the vertical angle GCF, and let DH be drawn parallel to E*r*, meeting C*r* in H. Then, by reaſon of the parallel lines, it will be as E*r* : DH (: : E*n* : D*n*) : : E*m* : DA; whence, E*r* being = E*m* (*by conſtruction*) DH and DA are alſo equal, and the point H falls in the periphery of the circle: therefore the angle *n*DH (*n*E*r*) at the center, ſtanding upon half the arch HC, will be equal to the angle HAC, at the periphery, ſtanding upon that whole arch, that is, equal to the difference of the angles ABC, and BAC; but the angle GFC being double to ABC, and FGC double to BAC (*by conſtruction*) the difference of GFC and FGC will be double to the difference between ABC and BAC, and therefore equal to 2*n*E*r* (2*n*DH) the difference given. Moreover, becauſe GCQ = FCQ, 2PCQ will be the difference between PCG and PCF, which muſt likewiſe be equal to 2*n*E*r*, the difference of their complements PGC and PFC; whence PCQ = *n*E*r*, and conſequently CQ = E*r*. Furthermore, ſince the angle ACG = CAG, and BCF = CBF, thence will CG = AG, and CF = FB; and therefore CG + GF + FC = AB. *Q. E. D.*

Method of calculation.

In the triangle E*nr* are given all the angles and the ſide E*r*, whence E*n* will be given; next, in the triangle AE*n* will be given (beſides the right-angle) both the legs E*n* and EA, whence the angle E*n*A is given; then it will be, as the radius to the ſine of DH*n* or E*rn* (1 : DH : D*n* : : DA : D*n*) ſo is the ſine of D*n*A to the ſine of DA*n*, whence AD*n*, the ſupplement of ACB, is alſo given; from which all the reſt of the angles in the figure are given by addition and ſubtraction only.

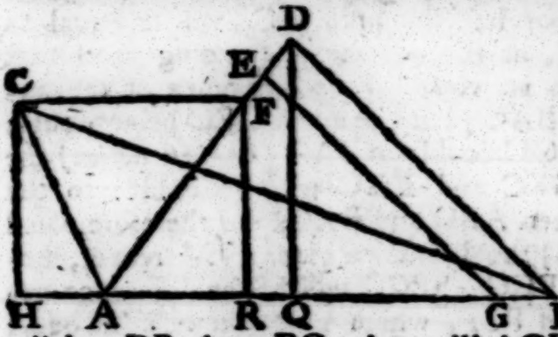
This method of ſolving the problem, it may be obſerved, requires three operations by the ſines and tangents, but the ſame thing may be performed by two proportions only: for as E*r* : AE : : the ſecant of *r*E*n* to the tangent of E*n*A; whence all the reſt will be found as above.

PROBLEM LXII.

To reduce a given triangle into the form of another, or to make a triangle which ſhall be ſimilar to one triangle, and equal to another.

CONSTRUCTION.

Upon the baſe AB of the triangle ABC, to which you would make another triangle equal, deſcribe ADB ſimilar to the triangle required; draw CF parallel to AB, meeting AD in F; take AE a mean proportional between AD and AF; and, parallel to DB, draw EG; then will AGE be the triangle that was to be conſtructed.



DEMONSTRATION.

Let FR and DQ be perpendicular to AB; then the triang. ADB : triang. ACB : : DQ : FR (*Schol. Euc.* 1. 6.) : : AD : AF (*Euc.* 4. 6.) : : AD^2 : AD × AF (*Euc.* 1. 6.) : : AD^2 : AE^2 (*by conſtruction*) : : triang. ADB triang. AEG (*Euc.* 19. 6.) Therefore, the antecedents of the firſt and laſt of theſe equal ratios being the ſame, the conſequents ACB and AEG muſt neceſſarily be equal. Q. E. D.

Method of calculation.

In the triangle ADB are given all the angles and the ſide AB, whence AD will be given; next, in the triangle AFR will be given all the angles and the ſide

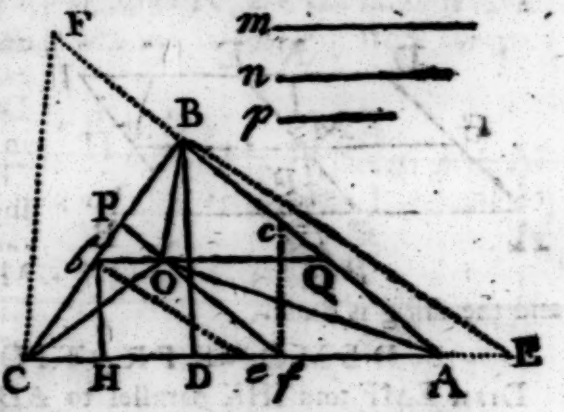
FR (= CH) whence AF will be given; and then, AD and AF being given, AE $= \sqrt{AD \times AF}$ will alſo be given.

PROBLEM LXIII.

To find a point in a given triangle ABC, *from whence right-lines drawn to the three angular points, ſhall divide the whole triangle into parts* (COA, AOB, BOC) *having the ſame ratio one to another, as three given right-lines,* m, n, *and* p, *reſpectively.*

CONSTRUCTION.

In CA and AB produced, if need be, take CE and AF, each equal to $m + n + p$, joining E, B and F, C; take Ce $= m$, Ac $= n$, and draw eb and cf, parallel to EB and CF, meeting the ſides of the given triangle in b and f; draw alſo bQ and fP parallel to AC and AB, and at O, the interſection of theſe lines, will be the point required.



DEMONSTRATION.

Let bH and BD be perpendicular to AC. The triangles CBE, Cbe, as alſo CBD, CbH are ſimilar; therefore, m (Ce) : $m + n + p$ (CE) : : Cb : CB : : bH : BD : : the triangle AOC : triangle ABC. In the very ſame manner it may be proved, that the part AOB is to the whole triangle ABC, as n to $m + n + p$; whence it follows, that the remaining part BOC muſt be to the whole triangle, as p to $m + n + p$; therefore theſe parts are to one another in the given ratio of m, n, and p. Q. E. D.

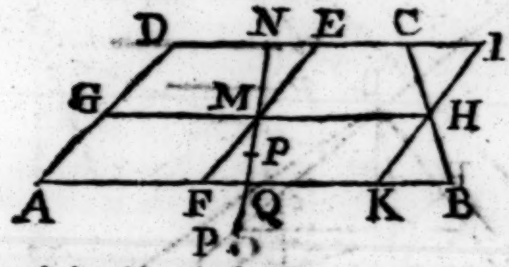
Method of calculation.

Since $\begin{cases} m+n+p : m :: AB : AQ, \\ m+n+p : n :: AC : Af\,(QO), \end{cases}$ both AQ and QO will be given from thence; then, in the triangle AOQ, will be given two ſides and the included angle, from which every thing elſe will be known.

PROBLEM LXIV.

To divide a given trapezium ABCD, *whoſe oppoſite ſides* AB, CD *are parallel, according to a given ratio, by a right-line* QN, *paſſing through a given point* P, *and falling upon the two parallel ſides.*

CONSTRUCTION.



Biſect AD in G, and draw GH parallel to AB (or DC) meeting BC in H; then divide GH in M, according to the given ratio, and through M draw PQN, and the thing is done.

DEMONSTRATION.

Draw EMF and IHK parallel to AD, meeting DC and AB in E, I, K and F.

Becauſe of the parallel lines, we have GD = ME = HI, and AG = FM = KH; whence, as GD is = AG (*by conſtruction*) ME will be = FM, and HI = HK; and the triangle EMN will be = FMQ, and IHC = BHK (*Euc.* 4. 1.) whence it appears that the trapezium AQND is alſo equal to the parallelogram DF, and the trapezium QBCN equal to the parallelogram FI; but theſe parallelograms are to one another as their baſes, or as GM to MH (*Euc.* 1. 6.); therefore GM : MH : : AQND : QBCN. *Q. E. D.*

Method of calculation.

Whereas AB and DC are parallel, GH is an arithmetical mean between them, and therefore equal to half their ſum. Therefore, as the whole line GH and the ratio of its parts GM, MH are given, the parts themſelves will alſo be given.

PROBLEM LXV.

To cut off from a given trapezium ABCD, *whoſe oppoſite ſides* AB, CD, *are parallel, a part* AQND *equal to a rectangle given, by a right-line paſſing through a given point* P, *and falling upon the two parallel ſides. (See the figure to the laſt problem.)*

CONSTRUCTION.

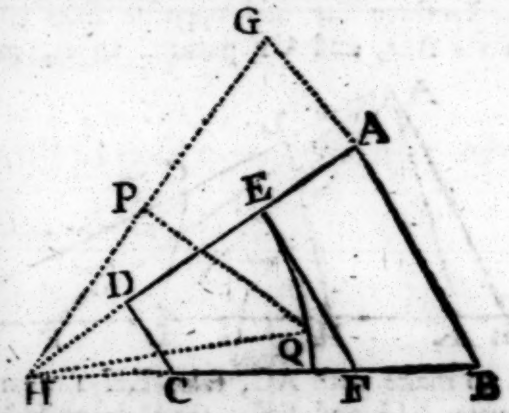
Biſect AD in G, and draw GH parallel to AB; upon AD (*by Euc.* 45. 1.) deſcribe the parallelogram ADEF equal to the rectangle given, and through the interſection of GH and EF draw PQN, and the thing is done: The demonſtration whereof is manifeſt from the preceding problem.

PROBLEM LXVI.

To divide a given trapezium ABCD, *whoſe ſides* AB *and* DC *are parallel, into two equal parts, by a right-line parallel to thoſe ſides.*

CONSTRUCTION.

Produce AD and BC till they meet in H, and make AG equal, and perpendicular to HD; draw HG and biſect the ſame with the perpendicular PQ = HP; join H, Q, and in HA take HE, equal to HQ, and parallel to AB draw EF, and the thing is done.



DEMONSTRATION.

Since $HE^2 \left(= HQ^2 = HP^2 + PQ^2 = 2HP^2 = \frac{HG^2}{2} = \frac{HA^2 + AG^2}{2} = \frac{HA^2 + HD^2}{2}\right)$ is an arithmetical-mean between HA^2 and HD^2, it is evident that the triangle HEF will alſo be an arithmetical-mean between the triangles HAB and HDC (or ABFE = EFCD); becauſe thoſe triangles, being ſimilar, are to one another as (HE^2, HA^2, HD^2) the ſquares of their homologous ſides. Q. E. D.

Method of calculation.

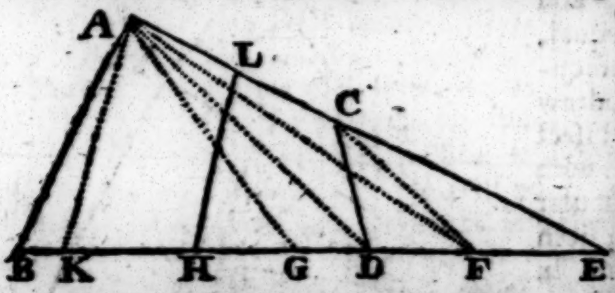
Since all the ſides and angles of the trapezium are ſuppoſed given, the ſide CD and all the angles of the triangle HDC will be given; therefore HD and AH will be known; whence HE, $= \sqrt{\frac{HD^2 + HA^2}{2}}$, will alſo be given. But the ſame thing may be had without the angles; for, ſince DC is parallel to AB, we have AB — DC : AD : : DC : HD; whence HE will be given, as before.

PROBLEM LXVII.

To divide a given trapezium ABCD *according to a given ratio, by a right-line* LH *cutting the oppoſite ſides* AC, BD *in given angles.*

CONSTRUCTION.

Produce the ſaid oppoſite ſides till they meet in E; draw AD, and CF parallel to it, meeting BE in F; divide BF in G, according to the given ratio; and, having made EAK equal to the given angle which LH is to make with AC, take EH a mean-proportional between EG and EK; then draw HL parallel to AK, and the thing is done.



DEMONSTRATION.

By conſtruction, EG : EH : : EH : EK : : EL : EA (*Euc.* 5. 6.) whence it follows that EG × EA =

$EH \times EL$, and conſequently that the triangles EHL and EAG are alſo equal to each other (*Euc.* 15. 6.) from which taking away EDC, common, the remainders CDHL and CDGA will be equal likewiſe, and conſequently $ALHB = AGB$, being the differences between thoſe remainders and ACDB. But the triangle ADF is $= ACD$, ſtanding upon the ſame baſe AD and between the ſame parallels; therefore (by adding AGD, common) AGF is alſo $= CDGA$ ($= CDHL$); but $AGF\ (CDHL) : AGB\ (ALHB) :: GF : GB$ (*Euc.* 1. 6.) Q. E. D.

Method of calculation.

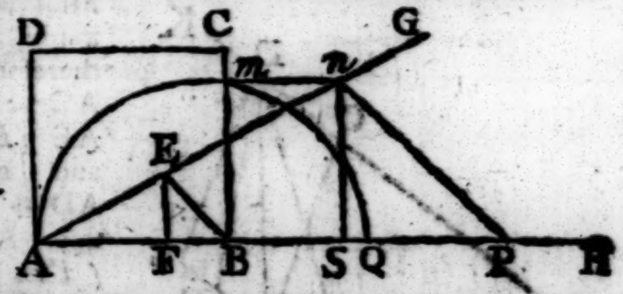
In the triangles ABE and ABK are given all the angles and the ſide AB, whence BE, BK, and EC will be known; then, in the triangle EFC will be given all the angles and the ſide CE, whence EF, and from thence FG and EG, will be known; laſtly, from the known values of EK, EG, and EF, the value of FH $(= \sqrt{EG \times EK} - EF)$ will be found.

PROBLEM LXVIII.

Two right-lines AG *and* AH, *meeting in a point* A, *being given by poſition; it is required to draw a right-line* nP *to cut thoſe lines in given angles, ſo that the triangle* AnP, *formed from thence, may be equal to a given ſquare* ABCD.

CONSTRUCTION.

Let the angle ABE be equal to the given angle APn, and let BE meet AG in E; draw EF perpendicular to AH, make BQ equal 2EF, and upon AQ deſcribe the ſemi-circle AmQ, cutting BC in m; draw mn parallel to AH, meeting AG in n, and nP parallel to EB, and AnP will be the triangle required.



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DEMONSTRATION.

The triangles AEB and A*n*P, being ſimilar, are to one another as the ſquares of their perpendicular heights EF and *m*B (*n*S) : but mB^2 is $=$ BQ × AB $=$ 2EF × AB; therefore it will be, as the triangle AEB (EF × $\frac{1}{2}$AB) : the triangle A*n*P : : EF^2 : 2EF × AB : : EF : 2AB : : EF × $\frac{1}{2}$AB : AB^2 (*Euc.* 1. 6.) wherefore, the antecedents being the ſame, the conſequents muſt neceſſarily be equal, that is, A*n*P $=$ ABCD. *Q. E. D.*

Method of calculation.

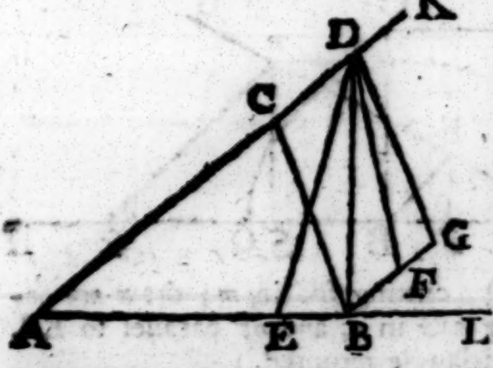
In the triangle ABE are given all the angles and the ſide AB, whence EF will be given, and conſequently S*n* ($= \sqrt{AB \times 2EF}$); whence AP and A*n* are alſo given.

LEMMA.

If from any point C, *in one ſide of a plane angle* KAL, *a right-line* CB *be drawn, cutting both ſides* AK, AL *in equal angles* (ACB, ABC); *and from any other point* D *in the ſame ſide* AK *another right-line be drawn, to cut off an area* ADE *equal to the area* ABC; *I ſay, that* DE *will be greater than* CB.

DEMONSTRATION.

Compleat the parallelogram DCBG, and join B, D, and in BG (produced if need be) take BF $=$ BE, and draw FD.



Since the triangles ABC, AED are equal, by ſuppoſition, and have one angle, A, common, therefore will AD : AC : : AB (AC) : AE (*Euc.* 15. 6.) and conſequently AD + AE greater than AC + AB (*Euc.* 25. 5.) whence it is manifeſt that CD muſt be greater than EB, or BG than BF. Moreover, becauſe the angle ABC (=

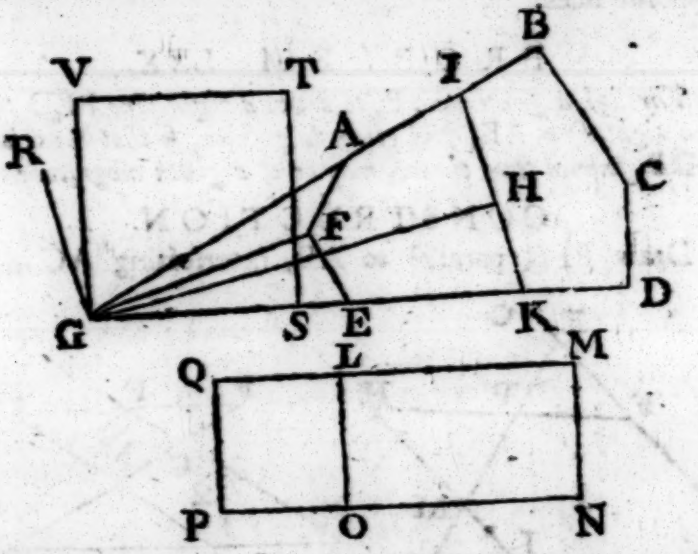
(= ACB = CDG) is = GBC, it will be greater than GBD, which is but a part of GBC; and therefore ABD muſt, evidently, be greater than GBD; wherefore, ſeeing BF and BE are equal, and that DB is common to both the triangles DBE, DBF, it is manifeſt that DE is greater than DF (*Euc.* 19. 1.); but DF is greater than DG (*by the ſame*) becauſe the angle DGF (DCB), being obtuſe, is greater than GFD, which muſt be acute (*Euc.* 32. 1.): conſequently DE is greater than DG, or its equal CB. *Q. E. D.*

PROBLEM LXIX.

From a given polygon ABCDEF, *to cut off a given area* AFEIK *by the ſhorteſt right-line*, KI, *poſſible.*

CONSTRUCTION.

Let the given area to be cut off be repreſented by the rectangle LMNO; and let the ſides AB and DE, by which the dividing line is terminated, be produced till they meet in G; make upon OL (*by Euc.* 45. 1.) a rectangle OQ equal to AFEG, and let a ſquare GSTV



be conſtituted (*by Euc.* 14. 2.) equal to the whole rectangle QN: biſect the angle BGD by the right-line GH, and make GR perpendicular to GH; and draw

KI, by the laſt problem, parallel to RG, ſo as to form the triangle KGI equal to the ſquare GSTV, and the thing is done.

DEMONSTRATION.

Since, by conſtruction, KGI (= GSTV) = QN, let AFEG = OQ be taken away, and there will remain AFEIK = LN. Moreover, ſince the angle HGI is = HGK, and the angle IHG (HGR) a right one, the angles I and K are equal; and therefore, by the preceding lemma, IK is the ſhorteſt right-line that can poſſibly be drawn to cut off the ſame area. Q. E. D.

Method of calculation.

Let the area of the figure AFEG be found, by dividing it into triangles AFG, EFG, and let this area be added to the given area to be cut off; and then, the ſquare root of the ſum being extracted, you will have GS the ſide of the ſquare GT; from whence GI will be determined, as in the laſt problem.

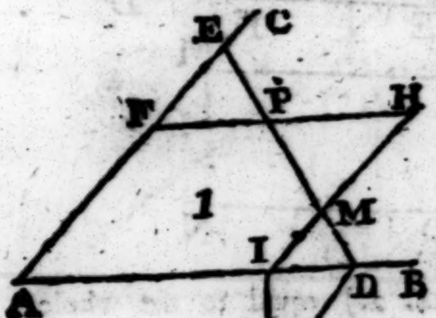
Note. In the ſame manner may a given area be cut off, by a right-line making any given angles with the oppoſite ſides.

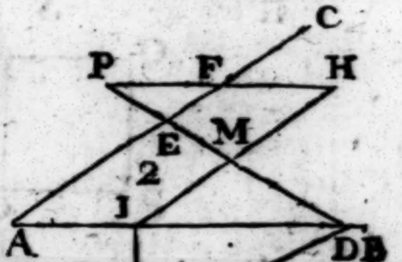
PROBLEM LXX.

Through a given point P, *to draw a right-line* PED *to cut two right-lines* AB, AC *given by poſition, ſo that the triangle* ADE, *formed from thence, may be of a given magnitude.*

CONSTRUCTION.

Draw PFH parallel to AB, interſecting AC in F;





and upon AF let a parallelogram AFHI be conſtituted equal to the given area of the triangle; make IK perpendicular to AI, and equal to FP; and, from the point K, to AB, apply KD = PH; then draw DPE, and the thing is done.

DEMONSTRATION.

Suppoſing M to be the interſection of DE and IH, it is evident, becauſe of the parallel lines, that all the three triangles PHM, PFE, and MDI are equiangular; therefore, all equiangular triangles being in proportion as the ſquares of their homologous ſides, and the ſum of the ſquares of PF (IK) and DI being equal to the ſquare of PH (KD), *by conſtruction and Euc.* 47. 1. it is evident that the ſum of the triangles PFE and DMI is = the triangle PHM; to which equal quantities *in fig.* 1, let AFPMI be added, ſo ſhall ADE be likewiſe equal to AFHI: but, *in fig.* 2, let PFE be taken from PHM, and there will remain EFHM = DMI; to which adding AIME, we have AFHI = ADE, *as before.* Q. E. D.

Method of calculation.

By dividing the given area by the given height of the point P above AB, the baſe AI of the parallelogram AFHI will be known, and conſequently PH (= KD); whence DI ($= \sqrt{KD^2 - PF^2}$) will likewiſe become known.—This problem, it may be obſerved, becomes impoſſible when KD (PH) is leſs than KI (PF); which can only happen, *in caſe* 1. when the given area is leſs than a parallelogram under AF and FP.

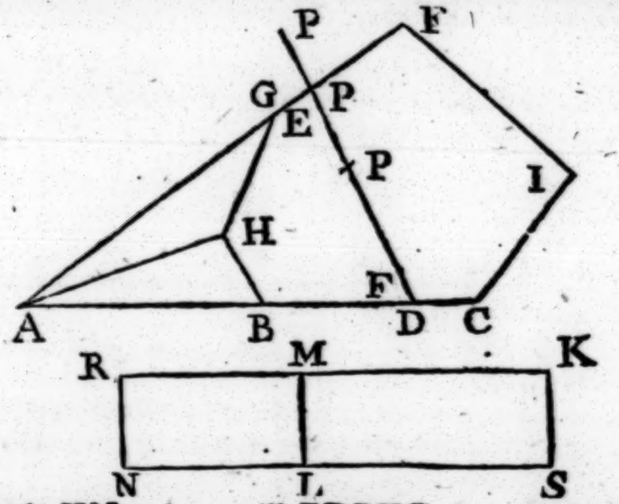
PROBLEM LXXI.

To cut off from a given polygon BCIFGH, *a part* EDBHG *equal to a given rectangle* KL, *by a right-line* ED *paſſing through a given point* P.

CONSTRUCTION.

Let the ſides of the polygon CB and FG, which the dividing line ED falls upon, be produced till they meet in A; upon ML (*by Euc.* 45. 1.) make the rectangle

MN equal to AGHB, and, by the laſt problem, let ED be ſo drawn through the given point P, that the triangle AED, formed from thence, may be equal to the whole



rectangle KN; then will EDBHG be equal to KL: for ſince AED is = KN, let the equal quantities AGHB and MN be taken away, and there will remain EDBHG = KL.

Method of calculation.

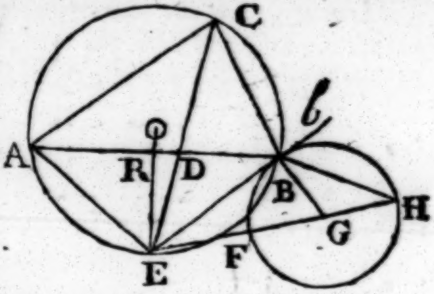
Let the area of the figure AGHB be found, by dividing it into triangles, and let this area be added to the area given, and the ſum will be equal to the area AED, or the rectangle KN; from whence AD will be found, as in the laſt problem.

PROBLEM LXXII.

Having the baſe, the vertical angle, and the length of the line biſecting that angle and terminating in the baſe, to deſcribe the triangle.

CONSTRUCTION.

Upon the given baſe AB let a ſegment of a circle ACB be deſcribed (*by Problem* 4.) to contain the given angle, and, having completed the whole circle, from O, the center thereof, perpendicular to AB, let the radius OE be drawn; draw EB, and make BG perpendicular thereto, and equal to half the given biſecting line; and from G, as a center, with the radius GB, let a circle BHF be deſcribed, interſecting EG (when drawn) in F and H; from E to AB draw ED = EF, and let the ſame be produced to meet the circumference in C; join A, C, and B, C; ſo ſhall ABC be the triangle required.



DEMONSTRATION.

The triangles CBE and BDE are ſimilar, becauſe the angle BEC is common to both, and the angles BCE and DBE ſtand upon equal arches BE and AE; therefore EC : EB : : EB : ED, and conſequently $ED \times EC = EB^2$: but (*by Euc.* 36. 3.) $EB^2 = EF \times EH = ED \times EH$ (*by conſtruction*). Hence $ED \times EC = ED \times EH$, and conſequently $EC = EH$; from which taking away the equal quantities ED and EF, there remains $DC = FH =$ the given line biſecting the vertical angle (*by conſtruction*): and it is evident that DC biſects the angle ACB, ſince ACD and BCD ſtand upon equal arches AE and EB. *Q. E. D.*

Method of calculation.

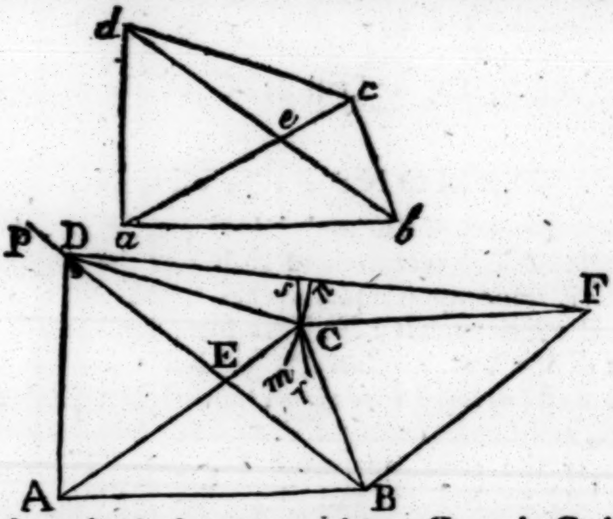
If BE be conſidered as a radius, BR ($\frac{1}{2}$AB) will be the co-ſine of the angle EBR, and BG the tangent of BEG; therefore BR : BG (or AB : DC) : : co-ſin. EBR (ACE) : tang. BEG, whoſe half-complement EHB is likewiſe given from hence: then, the angle HB*b* (ſuppoſing EB produced to *b*) being the complement of EHB, we ſhall have tang. EHB : rad. (: : ſin. EHB : co-ſin. EHB : : BE : EH : : EB : EC) : : ſin. ECB : ſin. CBE = ſin. EDB = co-ſin. OED, half the difference of the angles (ABC and BAC) at the baſe.

PROBLEM LXXIII.

Having given the two oppoſite ſides ab, cd, the two diagonals ac, bd, and alſo the angle aeb in which they interſect each other; to deſcribe the trapezium.

CONSTRUCTION.

In the indefinite line, BP, take BD equal to *bd*, and make the angle DBF equal to the given angle *aeb*, and BF = *ac*; alſo from the centers D and F, with the radii *dc* and *ab*, let two arches *m*C*n* and *r*C*s* be deſcribed interſecting each other in C; join D, C and F, C, and make BA equal and parallel to FC; then draw AD, AC, and BC, and the thing is done.



DEMONSTRATION.

Since *(by conſtruction)* AB is equal and parallel to CF, therefore will AC be equal and parallel to BF *(Euc.* 33. 1.) and conſequently the angle AEB *(Euc.* 29. 1.) = DBF = *aeb*. Q. E. D.

Method of calculation.

Join D, F; then in the triangle DBF will be given two ſides DB, BF and the angle included, whence the angle BFD and the ſide DF will be known; then in the triangle DFC will be given all the three ſides, whence the angle DFC will be known, from which BFC (BFD — DFC) = BAC will alſo be known.

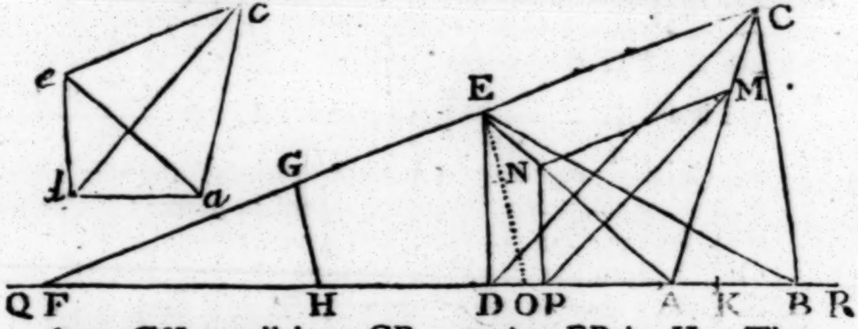
PRO-

PROBLEM LXXIV.

Having given the two diagonals and all the angles, to deſcribe the trapezium.

CONSTRUCTION.

Aſſume AB at pleaſure; and, having produced the ſame both ways, make the angles QAC, RBC equal, reſpectively, to two oppoſite angles *a* and *e* of the trapezium; moreover, make AÇF equal to *ace*, one of the remaining angles; and from F, the interſection of AF and BQ, take FG = the given diagonal *dc*, and draw GH parallel to CB, meeting FB in H. Then from A and B (*by the lem. p.* 334.) let two lines AE and BE be drawn to meet in FC, ſo as to be in the given ratio of *ae* to FH: in AE take AN = *ae*, and draw NM parallel to FC, meeting AC in M; laſtly, draw NP making the angle MNP = *ced*, and meeting FB in P; ſo ſhall AMNP be the true figure required.



DEMONSTRATION.

Let ED be parallel to NP, and let DC and PM be drawn.

It is evident, by conſtruction, that the diagonal AN, and all the angles of the trapezium, are equal to the reſpective given ones; it therefore remains only to prove that PM is equal to the other given diagonal *dc*. Now, the angle RBC being = CED (*by conſtruction*), the circumference of a circle may be deſcribed through all the four angular points of the trapezium BCED; and ſo the triangles FBE and FCD (as both the angles FBE and FCD ſtand upon the ſame chord ED) will be ſimilar; and conſequently BE : DC (: : FB : FC) : : FH : FG *(dc)*. But *(by conſtruction)* AE : BE : : *ae* : FH; therefore, by compounding theſe two proportions, we have AE : DC : : *ae* : *dc*; but (becauſe of the ſimilar figures ADEC, APNM) we alſo have AE : DC : : AN *(ae)* : PM; and conſequently PM = *dc*. Q. E. D.

Method of calculation.

All the angles of the triangles ABC, FAC, and FBC being given, we ſhall have ſin. ACB × ſin. F : ſin. ABC × ſin. ACF : : AB : AF; and ſin. FHG (FBC) : ſin. FGH (FCB) : : FG *(dc)* : FH; whence AF and FH are known.

Find $AK = \frac{AB \times ae}{FH + ae}$, and $KO = \frac{AK \times FH}{FH - ae}$ which laſt is equal to (OE) the radius of the circle determining the point E *(ſee the aforeſaid lemma)*. Therefore, in the triangle FOE are given two ſides FO and OE, beſides the angle F, whence the angle FOE will be given; then in the triangle AOE will be given OA, OE and the included angle; whence the angle OAE, which the diagonal AN makes with the ſide AP, will be known, and from thence every thing elſe required.

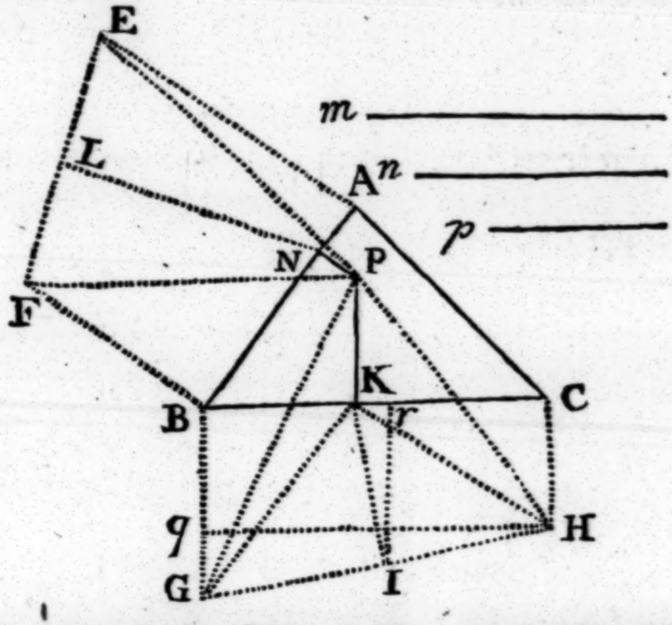
This problem, as the circle deſcribed from O cuts FC in two points, admits of two different ſolutions (except, only, when FC touches the circle). If the circle neither cuts nor touches that line, the problem will be impoſſible; the limits of the ratio of AE to BE (and conſequently of *ae* to *dc*) growing narrower and narrower, as AB becomes leſs and leſs, with reſpect to AC, or according as the ſum of the oppoſite angles $(a + e = QAC + RBC)$ approaches nearer and nearer to two right-angles; ſo that, at laſt (ſuppoſing AC and BC to coincide) AE and BE will be, *every-where*, in the ratio of equality; therefore *cd* can *here* have only one particular ratio to *ae*; and the diagonal ANE may be drawn at pleaſure, the problem being, in this caſe, indeterminate.

PROBLEM LXXV.

Suppoſing the right-lines m, n, p, to repreſent the lengths of three ſtaves erected perpendicular to the horizon, in the given points A, B, C; *to find a point* P, *in the plane of the horizon* ABC, *equally remote from the top of each ſtaff.*

CONSTRUCTION.

Join A, B and B, C, and make AE and BF perpendicular to AB; alſo make BG and CH perpendicular to BC, and let AE be taken $= m$, CH $= p$, and BF and BG each $= n$; draw EF and GH, which biſect by the perpendiculars LN and IK, cutting AB and CB in N and K; make KP and NP perpendicular to BC and BA, and the interſection P of thoſe perpendiculars will be the point required.



DEMONSTRATION.

Conceive the planes AEFB and BCHG to be turned up, ſo as to ſtand perpendicular to the plane of the horizon ABC and interſect it in the right lines AB and BC; then, becauſe BF and BG are equal to each other, and perpendicular to the plane of the horizon, it is evident that the points F and G muſt coincide, and that AE, BG (BF) and CH will repreſent the true poſition of the ſtaves: ſuppoſe KG, KH, PG, PH, PE, and PF to be now drawn; then, ſince *(by conſtruction)* GI = HI, and the angle GIK = HIK, therefore is GK = HK *(Euc.* 4. 1.): moreover, ſince KP is *(by conſtruction)* perpendicular to BC, it will alſo be perpendicular to the plane BCHG, and conſequently the angles PKG and PKH both right-angles: therefore, ſeeing the two triangles GKP, HKP have two ſides and an included angle equal, the remaining ſides PG and PH muſt likewiſe be equal *(Euc.* 4. 1). After the very ſame manner it is proved that PF (or PG) is equal to EP. *Q. E. D.*

Method of Calculation.

Draw I*r* perpendicular, and H*q* parallel to BC; then, by reaſon of the ſimilar triangles H*q*G and I*r*K, it will be as BC (Hq) : BG — CH (Gq) : : $\frac{BG + CH}{2}$ (Ir) : $Kr = \frac{\overline{BG - CH} \times \overline{BG + CH}}{2BC}$; which ſubtracted from Br ($=\frac{1}{2}$BC) gives BK: and, in the ſame manner will BN be found; then in the trapezium KBNP will be given all the angles and the two ſides BK and BN; from whence the remaining ſides, &c. may be eaſily determined.

PROBLEM LXXVI.

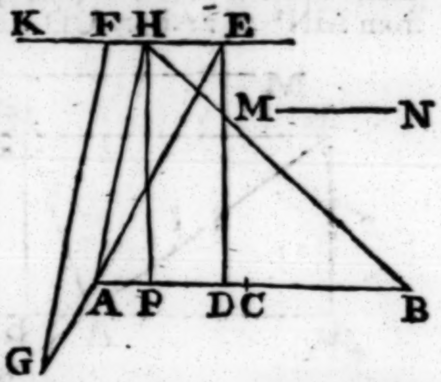
The baſe, the perpendicular, and the difference of the ſides being given, to determine the triangle.

CONSTRUCTION.

Biſect the baſe AB in C, and in it take CD a third proportional to 2AB and the given difference of the ſides MN; erect DE equal to the given perpendicular, and draw EK parallel to AB, and take therein EF = MN; draw EAG, to which, from F, apply FG = AB; draw AH parallel to FG meeting EK in H; then draw BH, and the thing is done.

DEMONSTRATION.

By reaſon of the parallel lines, FG (AB) : FE (MN) :: AH : EH (DP); therefore $AB \times DP = AH \times MN$, or $2AB \times DP = 2AH \times MN$; to which laſt equal quantities adding $2AB \times CD = MN^2$ *(by conſtruction)* we have $2AB \times CP = 2AH \times MN + MN^2$; but $2AB \times CP$ is $= BH^2 - AH^2$ *(by a known property of triangles)*; therefore $BH^2 - AH^2 = 2AH \times MN + MN^2$, or $BH^2 = AH^2 + 2AH \times MN + MN^2 = \overline{AH + MN}|^2$ *(Euc. 4. 2.)* conſequently $BH = AH + MN$. *Q.E.D.*



Method of calculation.

In the right-angled triangle ADE we have DE and AD $\left(= \frac{1}{2}AB - \frac{MN^2}{2AB}\right)$ whence the angle DAE (FEG) will be found; then in the triangle EFG will be given two ſides and one angle, from which the angle GFK (= BAH) will alſo be known.

PROBLEM LXXVII.

The baſe, the perpendicular, and the ſum of the two ſides being given, to deſcribe the triangle.

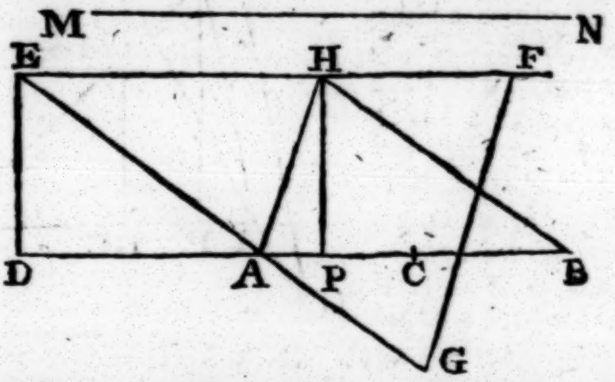
CONSTRUCTION.

Biſect the baſe AB in C, and in it produced take CD a third proportional to 2AB and the ſum of the ſides, MN; erect DE equal to the given perpendicular, and draw HE parallel to AB, and take therein EF = MN; draw EAG, to which from F, apply FG = AB, draw AH parallel to FG, meeting EF in H, then draw BH, and the thing is done.

DEMON-

DEMONSTRATION.

Becauſe of the parallel lines, FG (AB) : FE (MN) : : AH : EH (DP) ; and therefore $2MN \times AH = 2AB \times DP$; which equal quantıes being ſubtracted from $MN^2 = 2AB \times CD$ *(by conſtruction)* there will



remain $MN^2 - 2MN \times AH = 2AB \times CP = BH^2 - AH^2$; whence, by adding AH^2 to each, we have $MN^2 - 2MN \times AH + AH^2 = BH^2$, that is, $\overline{MN - AH}|^2 = BH^2$; therefore $MN - AH = BH$, or $MN = BH + AH$. *Q. E. D.*

Method of calculation.

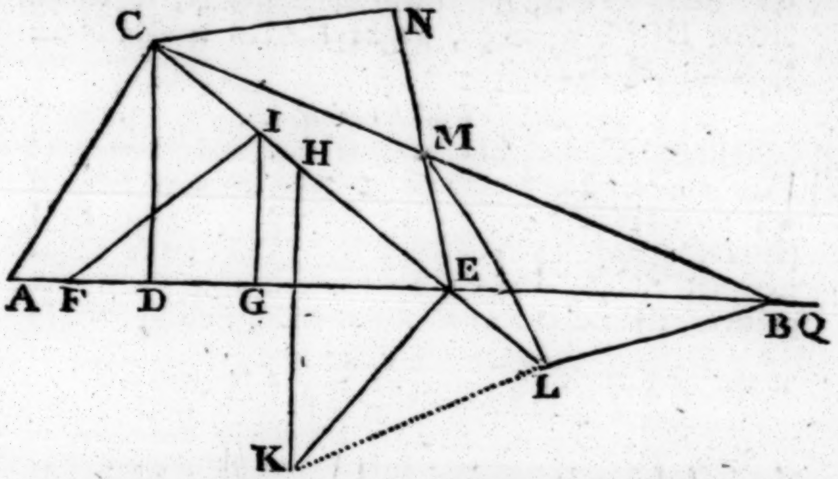
In the triangle AED are given (beſides the right-angle) both the legs, whence the angle DAE (= FEG) will be given ; then in the triangle FEG one angle and two ſides will be known, from which the angle EFG (= BAH) will be determined.

PROBLEM LXXVIII.

The difference of the two ſides, the perpendicular, and the vertical angle being given, to determine the triangle.

CONSTRUCTION.

Upon the indefinite line FEQ erect the given perpendicular DC, making the angle DCE = half the given angle ; let EF, expreſſing the given difference of the ſides, be biſected by the perpendicular GI, meeting EC in I ; alſo let EC be biſected in H, and make EK perpendicular to CE, and equal to EI ; and having drawn HK, take HL, in HE produced, equal thereto; from L to FQ apply LB = EK, and join C, B; alſo



draw EM, making the angle CEM equal to DEC, and cutting CB in M; then from C to QEF apply CA = CM, ſo ſhall ACB be the triangle required.

DEMONSTRATION.

Upon EM let fall the perpendicular CN, and join L, M and F, I. Now $LB^2 = EK^2$ *(by conſtruction)* $= \overline{HK + HE} \times \overline{HK - HE}$ *(Euc.* 5. 2.) $= \overline{HL + CH} \times \overline{HL - HE}$ *(by conſtruction)* = CL × EL; whence CL : LB : : LB : EL; therefore the triangles CLB and ELB muſt be equiangular *(Euc.* 6. 6.) and conſequently LBM = LEB = CED = CEM *(by conſtruction)*. Therefore, ſince the external angle CEM of the trapezium LEMB, is equal to the oppoſite internal angle B, the circumference of a circle will paſs through all the four angular points; and conſequently the angle LMB will be = LEB, both ſtanding upon the ſame chord LB; but it is proved that LBM is = LEB, therefore LMB = LBM = FEI; and ſo the triangles BLM and EIF, being iſoſceles, and having LMB = EFI, and alſo LB = EI *(by conſtruction)* they will be equal in all reſpects, and conſequently BM = EF; whence BC — AC (= BC — CM = BM) = EF, the given difference (*by conſtruction*). Moreover, CEN being = CED (*by conſtruction*), CN will be = CD; and ſo CM being = CA, ACD will be = MCN, to which adding DCM, common, we have ACB = DCN = 2DCE. *Q. E. D.*

Method of calculation.

Seeing EG and EH are the ſine and tangent of EIG and EKH, to the equal radii EI and EK, it will therefore be EG : EH (or EF : EC) : : ſin. EIG (ECD) : tang. EKH. But, EC : CD : : the radius : co-ſin. ECD; whence, by compounding theſe proportions, EF : CD : : rad. × ſin. ECD : co-ſin. ECD

$$\times \text{ tang. EKH} : : \frac{\text{rad.} \times \text{ſin. ECD}}{\text{co-ſin. ECD}} (= \text{tangent ECD}) :$$

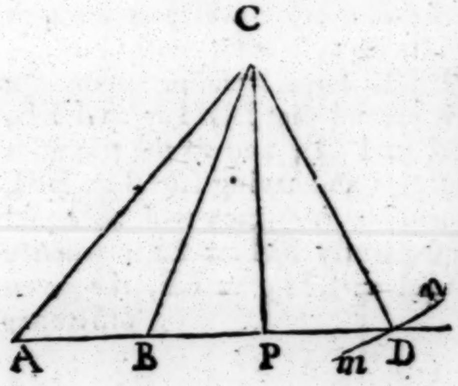
tang. EKH; from which EKL, half the complement of EKH will be alſo given: then it will be, as the radius : tang. EKL (: : KE : EL : : LB : EL) : : ſin. LEB (CED) : ſin. LBE (BCE); which proportions, expreſſed in words, give the following Theorem.

As the difference of the ſides is to the perpendicular, ſo is the tangent of half the vertical angle to the tangent of an angle; and as the radius is to the tangent of half the complement of this angle, ſo is the co-ſine of half the vertical angle to the ſine of half the difference of the angles at the baſe.

PROBLEM LXXIX.

The perpendicular, the difference of the ſides, and the difference of the angles at the baſe being given, to determine the triangle.

CONSTRUCTION.



Let a triangle ABC be conſtructed, *by the laſt problem*, whoſe perpendicular and difference of the ſides ſhall be the ſame with *thoſe* given, and whereof the vertical angle ACB is alſo equal to the given difference of angles:

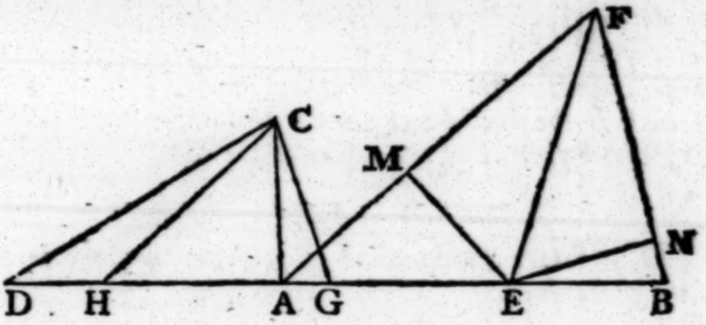
then upon C, as a center, with the radius CB let an arch be deſcribed, interſecting AB, produced, in D; join C, D, and ACD will be the triangle required. For CD being = CB, the angle CDB will alſo be = CBD = A + BCA (*Euc.* 32. 1). The method of calculation is alſo the ſame as in the preceding problem.

PROBLEM LXXX.

The perpendicular, the ſum of the two ſides, and the vertical angle being given; to deſcribe the triangle.

CONSTRUCTION.

Upon AB, the given ſum of the two ſides, erect AC equal to the given perpendicular; and make the angle ACH equal to the complement of half the given angle: upon AB (*by Prob.* 72.) let a triangle ABF be conſtituted, whoſe vertical angle AFB ſhall be equal to the given one, and whereof the biſecting line FE (terminating in the baſe) ſhall be = DC; then draw CG and GH parallel to FB and FA, ſo ſhall GCH be the triangle required.



DEMONSTRATION.

It is evident that the angle HCG is = AFB = the given one. Moreover, if EM and EN be taken as perpendiculars to AF and BF, they will be equal to each other, and alſo equal to the given one AC, becauſe all the angles EFN, EFM, and ADC are equal, by conſtruction, and EF is likewiſe = CD; whence, as the angles AHC, AGC are reſpectively equal to EAM, EBN, it is evident that HC = EA, and GC = EB, and conſequently, that HC + GC (= EA + EB) = AB. *Q. E. D.*

Method of calculation.

By the problem above referred to, AB : CD (EF) : : co-ſin. ADC (AFE) : tang. of an angle; which let be denoted by Q.

Now CD : CA : : rad. : ſin. ADC; which proportion being compounded with the former, we have AB : CA : : co-ſin. ADC × rad. : tang. Q × ſin. ADC : : $\frac{\text{co-ſin. ADC} \times \text{rad.}}{\text{ſin. ADC}}$ (co-tangent ADC) : tang. Q.

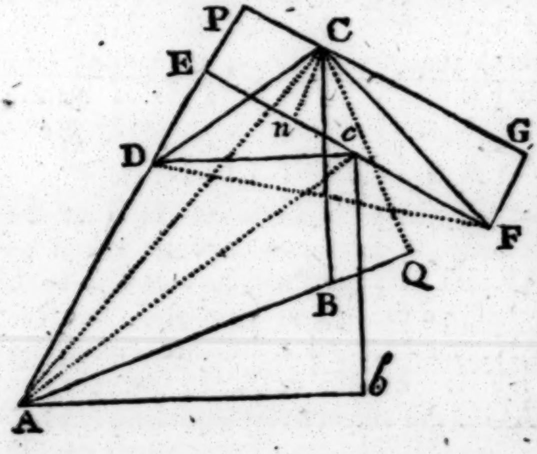
Then, by the ſame problem, it will be as tang. $\frac{1}{2}$Q : rad. : : ſin. ADC : co-ſin. of the difference of the angles (G and H) at the baſe. The above proportions, given in words at length, exhibit the following Theorem.

As the ſum of the ſides is to the perpendicular, ſo is the co-tangent of half the vertical angle to the tangent of an angle; and, as the tangent of half this angle is to the radius, ſo is the ſine of half the vertical angle to the co-ſine of half the difference of the angles at the baſe.

PROBLEM LXXXI.

To conſtitute a trapezium of a given magnitude under four given lines.

CONSTRUCTION.



Make a right-angle *b* with two of the given lines A*b*, *bc*; and with the other two complete the trapezium A*bc*D : upon AD let fall the perpendicular *c*E, in which produced (if neceſſary) take EF, ſo that the rectangle under it, and AD, may be double the given area: moreover, take a fourth-proportional to AD, Ab, and bc, with which, from the center F, let an arch be deſcribed, meeting another arch, deſcribed from D with the radius Dc, in C; join D, C; and from A and C draw the other two given lines AB, CB, ſo as to meet, and they will thereby form the trapezium ABCD, as required.

DEMONSTRATION.

Draw Ac, AC, and FC; upon AD and AB let fall the perpendiculars CP, CQ; and make FG perpendicular to PCG.

Becauſe $AD^2 + DC^2 + 2AD \times DP$ ($= AC^2$, *Euc.* 12. 2.) $= AB^2 + BC^2 + 2AB \times BQ$, and $AD^2 + Dc^2 + 2AD \times DE$ ($= Ac^2$) $= Ab^2 + bc^2$ (*Euc.* 47. 1.) it follows, by taking theſe laſt equal quantities from the former, that $2AD \times DP - 2AD \times DE$ ($2AD \times EP$) $= 2AB \times BQ$, and conſequently that BQ : EP (FG) : : AD : AB : : BC : FC (*by conſtruction*) whence the triangles BCQ, FCG are ſimilar, and ſo CQ : CG : : BC : FC : : AD : AB (*by conſtruction*) and therefore $CQ \times AB = CG \times AD$; hence, by adding $CP \times AD$ to each, we have $CP \times AD + CQ \times AB$ (= twice the area ABCD) $= CP \times AD + CG \times AD = EF \times AD$ = twice the given area (*by conſtruction*). Q. E. D.

Method of calculation.

From DE $\left(= \dfrac{Ab^2 + bc^2 - AD^2 - Dc^2}{2AD}\right)$ and EF $\left(= \dfrac{2 \text{ area}}{AD}\right)$ the value of DF, and likewiſe that of the angle ADF, will be found; then, all the ſides of the triangle DCF being known, the angle FDC will likewiſe be known; which, added to ADF, gives (ADC) one of the angles of the trapezium.

It may ſo happen that a trapezium, having one right angle, cannot be conſtituted under the four given lines; in which caſe it will be neceſſary (inſtead of forming the trapezium A$bc$D) to lay down AD firſt, and in it (produced if needful) to take DE equal to $\frac{\overline{AB}|^2+\overline{BC}|^2-\overline{AD}|^2-\overline{DC}|^2}{2AD}$, that is, equal to the altitude of a rectangle, formed on the baſe 2AD, whereof the contained area is equal to the difference of $\overline{AB}|^2+\overline{BC}|^2$ and $\overline{AD}|^2+\overline{DC}|^2$ (which line DE is to be ſet off on the other ſide of D, when the latter of theſe two quantities is the greater): this being done, the reſt of the ſolution will remain the ſame, as is manifeſt from the firſt and ſecond ſteps of the demonſtration; the proceſs from thence to the end being no-ways different.

It may be further obſerved, that the problem itſelf becomes impoſſible, when the two circles, deſcribed from the centers D and F, neither cut nor touch; the greateſt limit of the area, and conſequently of EF, being when they touch each other; in which caſe, the ſum of the radii DC, FC becoming = DF, the point C will fall in the line DF, and the angle DCF will become equal to two right angles: but the ſum of the oppoſite, external angles CDP and CBQ is always equal to DCF; becauſe CDP (ſuppoſing C*n* parallel to AP) is = DC*n*, and CBQ (= CFG) = FC*n*: hence it is evident that the limit, or the greateſt area will be when the ſum of the oppoſite angles is equal to two right-angles, or when the trapezium may be inſcribed in a circle.

FINIS.

20 JY 66